Arbitrageurs, bubbles, and credit conditions*

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December 2012

Abstract

We study a pure exchange economy populated by three types of agents: constrained agents who are subject to a risk constraint, unconstrained agents who are only subject to a standard nonnegative wealth constraint, and arbitrageurs who, in addition to being unconstrained, may incur transitory losses that are bounded by a state-dependent credit limit. This uncollateralized credit facility is valuable when there are bubbles, which arise endogenously due to the presence of constrained agents. Since arbitrageurs are required to hold less collateral than other agents, their presence implies a reduction in the value of the stock's collateral services and therefore a decrease in the relative size of the bubble. In contrast to previous results in the literature, we show that the presence of risky arbitrage trading has an impact on the stock price level and makes it more volatile than the underlying fundamental, thereby generating the leverage effect.

Keywords: Collateral value; Exchange economy; Leverage effect; Limits of arbitrage; Rational bubbles; Wealth constraints.

JEL Classification: D51, D52, G11, G12.

*We thank Hengjie Ai, Rui Albuquerque, Harjoat Bhamra, Jérôme Detemple, Bernard Dumas, Phil Dybvig, Marcel Rindisbacher, and seminar participants at the Bank of Canada, Boston University Macro Workshop, the 8th Cowles Conference on General Equilibrium and its Applications, the 2012 Financial Intermediation Research Society Conference, Universidad Adolfo Ibáñez, University of Geneva and University of Lugano for helpful comments. Financial support by the Swiss National Center of Competence in Research NCCR FinRisk (Project A5), the Swiss Finance Institute and Boston University is gratefully acknowledged.

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1 Introduction

The existence of asset pricing bubbles in equilibrium models with continuous trading is closely related to the question: if the security is overpriced, why is its price not corrected by short sales?. The answer has to do with trading frictions that limit the ability of investors to profit from the arbitrage opportunity that a bubble provides. Wealth constraints[1] are one important example of such frictions since they are effectively bounds on credit. These constraints remove downward selling pressure on mispriced assets by limiting the scale at which an investor can sustain a strategy that would exploit bubbles,[2] hence their economic significance. Our goal in this paper is to characterize the role of wealth constraints in an equilibrium model with bubbles. In particular, we explore when and how they improve the ability of investors to exploit arbitrage strategies and analyze the impact of these strategies on prices.

We answer these questions by building an exchange economy with endogenous mispricing that nests Hugonnier (2012). This model considers a riskless asset in zero net supply, a dividend paying risky asset in positive supply and two types of price-taking agents endowed with assets. Constrained agents are subject to portfolio constraints which force them to keep a long position in the riskless asset, while unconstrained agents are free to choose the composition of their portfolio. Both types of agents face a standard non-negativity constraint on wealth. In agreement with intuition, the interest rate decreases and the Sharpe ratio increases with respect to a frictionless economy, yet the stock and the riskless asset may contain bubbles in order to incite unconstrained agents to hold positions that are compatible with market clearing.[3]

We add a third class of investors: arbitrageurs. These agents do not face portfolio constraints and benefit from credit conditions that allow them to withstand transitory

[1]Wealth constraints were proposed by Harrison and Kreps (1979) and Dybvig and Huang (1988) as a mechanism to preclude doubling strategies. Doubling strategies are essentially sequences of bets that win for sure in finite time by doubling up after a loss. A model of continuous trading in which agents prefer more to less requires some constraint to make doubling strategies infeasible, for otherwise there can be no equilibrium. Wealth constraints are widely used in continuous-trade models, but are also introduced in discrete-time, infinite-horizon general equilibrium models (see e.g., Kocherlakota (1992), Magill and Quinzii (1994) among others).

[2]For example, an arbitrage strategy that exploits a bubble involves short selling the higher-cost asset and buying the lower-cost replicating portfolio, while investing the time zero proceeds during the convergence trade. This strategy requires no initial wealth and provides positive payoffs, but may not be feasible at all scales due to the presence of wealth constraints.

[3]The presence of bubbles in exchange economies with portfolio constraints is reminiscent of Caballero (2006) (see also Caballero and Krishnamurthy (2009)), who points to asset shortages and (portfolio) imbalances as a source of bubbles. These two features correspond in our model to fixed supply in securities and portfolio constraints, respectively.

marked-to-market losses, i.e., states with negative net worth. Trading in these states may be sustained through credit implicitly extended by the other agents in the economy.[4] Throughout, we assume all agents have logarithmic preferences.

In our main example, we find that the ability of arbitrageurs to exploit bubbles improve when credit depends on the size of the market, that is, when it increases in times where the stock market is high and dries up in bad times.[5] In particular, we show that arbitrageurs are able to reap arbitrage profits from the market and consume even if they do no have any initial capital. The impact of these strategies on equilibrium prices is summarized next.

First, the mispricing on both securities decreases with improved credit conditions, to the extent that the bubble on the stock vanishes in the limit of infinite credit. In particular, we show that as the size of the credit facility increases, the stock price level decreases due to the fact that the arbitrageur may hold a class of arbitrage strategies which require less collateral, and hence, in equilibrium, it implies a reduction in the value of collateral services provided by the stock. The reduction on the stock level also decreases the bubble on the riskless asset.[6]

Second, in contrast with pure exchange economies that feature multiple agents with homogenous logarithmic utility and frictions,[7] the stock volatility is higher than the volatility of dividends in all states. This follows from the fact that the volatility of the price dividend ratio, which determines the stock volatility in excess of the volatility of dividends, is positive as a result of the arbitrageur's trading activity. We also show that the volatility level is increasing in the size of the credit facility.

Third, the equity premium and the stock volatility are higher in bad times, generating the leverage effect, i.e., the well-established fact according to which volatility increases when the stock price falls. The model generates also a low interest rate. These results are of interest because they show how, by introducing heterogeneity in investors' credit conditions, a stylized model with logarithmic preferences and portfolio constraints may help in explaining empirical regularities.

[4]Unlike Shleifer and Vishny (1997), there are no outside investors or agency conflicts in the model, yet we build on the idea of the limited effectiveness of arbitrageurs in bringing prices closer to fundamental values due to credit frictions. See Gromb and Vayanos (2010) for a recent survey of this literature.

[5]We also present examples where an equilibrium may fail to exist or may be invariant to the composition of the credit facility.

[6]This result sheds light on Fahri and Tirole (2010) who question whether authorities could regulate the likelihood and the size of bubbles by relaxing securitization (or collateralizability) standards.

[7]See e.g., Detemple and Murthy (1997), Basak and Cuoco (1998) and Basak and Croitoru (2000) among others.

We also explore the impact of arbitrageurs on welfare. We show that both constrained and unconstrained agents can be made worse off, as the arbitrageur's trading activity affects prices in a way that may be unfavorable to them. In particular, improved credit conditions reduce both the unconstrained and constrained agents' initial wealth, if endowed with shares of the stock, and increase the stock volatility. The latter effect worsens the stock's 'collateral quality' and affects negatively the portfolio mix held by the unconstrained agent.

Our work is related to various strands of literature. There are few papers where arbitrage opportunities arise endogenously in equilibrium. For example, Gromb and Vayanos (2002) and Basak and Croitoru (2000) study economies where all agents are subject to portfolio constraints and, as a result, the ability of investors to benefit from the mispriced assets is limited. If these constraints are lifted for some agents, then these agents can scale their position to an arbitrary size and the presence of mispriced assets becomes inconsistent with the existence of an equilibrium. In addition, Gromb and Vayanos (2002) focus on the impact of arbitrageurs on welfare. They examine a different form of constraints which induce segmented markets. Without the arbitrageur, there is no trade across markets; in the presence of the arbitrageur, who acts as an intermediary by running riskless arbitrages, Pareto-improving trade occurs. In contrast, the arbitrageur in our model does not alleviate the portfolio constraints, quite the contrary, his trading activity may affect negatively the welfare of both unconstrained and constrained agents. Basak and Croitoru (2006) build a production economy version of Basak and Croitoru (2000), and introduce a risk neutral arbitrageur with position constraints and zero net wealth whose trading activity brings prices closer to their fundamental values through costless and riskless trades. In doing so, the arbitrageur always takes the maximum position allowed by the portfolio constraint. Our economy features a risk averse arbitrageur that also starts from a zero wealth position, however, the arbitrageur accumulates capital because not all arbitrage profits are consumed and importantly, his arbitrage trades are risky, in the sense that they may involve temporary losses prior to closure. Moreover, the production technology in Basak and Croitoru (2006) determines the stock price dynamics exogenously, rendering a flat stock volatility. Our model shows that risky arbitrage strategies have important implications on the price dynamics, which are fully endogenous.

The key contributions in the literature of equilibrium asset pricing bubbles in models with continuous trading have studied primarily nonnegativity constraints on wealth. Loewenstein and Willard (2000) show that, in complete-market frictionless economies, bubbles may exist on zero net supply securities, such as options and futures, but not on positive net supply securities such as stocks. Hugonnier (2012) shows that the presence of portfolio constraints may generate equilibrium pricing bubbles also on positive net supply securities. Prieto (2012) extends Hugonnier (2012) in a setting with heterogeneity in risk aversion and beliefs across agents. In addition to these papers, there are studies that analyze, mostly in partial equilibrium, the properties of asset pricing bubbles. Cox and Hobson (2005) and Heston, Loewenstein, and Willard (2007) study bubbles on the price of derivatives written on the stock and show that put-call parity might not hold. Jarrow, Protter, and Shimbo (2010) introduce regime shifts to show that a bubble on the stock can burst and be born in models with incomplete markets. An important difference with respect to these studies lies in the fact that they assume the existence of a risk neutral probability measure. This will not be the case in the economy analyzed in this paper, since the presence of a bubble on the riskless asset is equivalent to the non existence of a risk neutral probability measure.

The remainder of the paper is structured as follows. Section 2 present the main assumptions about the economy, the traded assets and the agents. Section 3 solves for the unique equilibrium in the economy. Section 4 discusses the main implications and insights of the model. Section 5 shows that similar asset pricing implications are obtained for an alternative credit facility and analyzes the implications of a sudden reversal in credit conditions. Section 6 concludes. All proofs are gathered in the Appendix.

2 The model

2.1 Information structure

We consider a continuous time economy on an infinite horizon and assume that the uncertainty in the economy is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that carries a standard Brownian motion Z_t. All random processes are assumed to be adapted with respect to the augmentation of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ generated by the Brownian motion, and all statements involving random quantities are understood to hold either almost surely or almost everywhere depending on the context.

2.2 Securities markets

Agents trade in two securities: a locally riskless savings account in zero net supply and one risky asset, or stock, in positive supply of one unit. The price of the riskless asset evolves according to

$$S_{0t} = 1 + \int_0^t S_{0u} r_u du$$

for some short rate process $r \in \mathbb{R}$ that is to be determined in equilibrium. On the other hand, the stock is a claim to a dividend process δ that evolves according to a geometric Brownian motion,

$$\delta_t = \delta_0 + \int_0^t \delta_u (\mu_\delta du + \sigma_\delta dZ_u)$$

for some constant drift μ_δ and some constant volatility $\sigma_\delta > 0$. The stock price process is denoted by S and evolves according to

$$S_t + \int_0^t \delta_u du = S_0 + \int_0^t S_u (\mu_u du + \sigma_u dZ_u)$$

for some initial value $S_0 > 0$, some drift $\mu \in \mathbb{R}$ and some volatility $\sigma \in \mathbb{R}$ processes which are to be determined in equilibrium.

2.3 Trading strategies

A trading strategy is a pair of processes $(\pi; \phi)$ where π represents the amount invested in the stock while ϕ represents the amount invested in the riskless asset. A trading strategy is said to be self-financing given initial wealth w and consumption rate c if the corresponding wealth process

$$W_t = W_t(\pi; \phi) \equiv \phi_t + \pi_t \tag{1}$$

satisfies the dynamic budget constraint

$$W_t = w + \int_0^t (\phi_u r_u + \pi_u \mu_u - c_u)\, du + \int_0^t \pi_u \sigma_u dZ_u. \tag{2}$$

Implicit in the definition is the requirement that the trading strategy be such that the above stochastic integrals are well-defined.

2.4 Agents

The economy is populated by three agents indexed by $k = 1, 2, 3$. The preferences of agent k are represented by

$$U_k(c) \equiv E\left[\int_0^\infty e^{-\rho t} \log(c_t) dt\right]$$

for some subjective discount rate $\rho > 0$ and we let $w_k \equiv \beta_k + \alpha_k S_0$ denote the initial wealth of agent k computed at equilibrium prices.

The three agents in the economy have homogenous preferences and beliefs but differ in their trading opportunities. Agent 1 is free to choose any self-financing strategy whose wealth is nonnegative at all times, and we will refer to him as the unconstrained agent. Agent 2, to whom we will refer as the constrained agent, is subject to the same requirement as agent 1, but must in addition choose a strategy that satisfies

$$\pi_t \in \mathcal{C}_t \equiv \{\pi \in \mathbb{R} : |\sigma_t \pi| \leq (1 - \varepsilon)\sigma_\delta W_t\},$$

for some fixed constant $\varepsilon \in [0, 1]$. This constraint can be thought of as a stock market participation constraint, that limits the amount of risk the agent can take while trading. In particular, if $\sigma_t \geq \sigma_\delta$ (which we will show is the case in equilibrium) then this constraint forces agent 2 to invest a strictly positive fraction of his wealth in the money market account and thereby introduces an imbalance that ultimately generates bubbles. This constraint is also a special case of the general risk constraints in Cuoco, He, and Isaenko (2008), and is also studied in Gârleanu and Pedersen (2007) and Prieto (2012) as a constraint on conditional value-at-risk.

Agent 3 is also free to choose any self-financing strategy but, in contrast to the two other agents, he is not required to maintain nonnegative wealth at all times. Instead, this agent is allowed to run short term deficits provided that

$$W_{3t} = \phi_{3t} + \pi_{3t} \geq -\psi S_t, \tag{3}$$

for some exogenously fixed constant $\psi \geq 0$. This agent should be thought of as an arbitrageur whose funding liquidity conditions are determined by the parameter ψ. The fact that the amount of credit available to this arbitrageur increases with the size of the market captures in a simple way the observation that liquidity improves in times where the stock market is high and dries up in bad times.

Since agent 3 can continue trading in states of negative wealth, the solvency constraint (3) allows for excess borrowing. Trades in these states may be considered uncollateralized as agent 3 does not have enough assets to cover his liabilities in case of instantaneous liquidation. Note however that, given the assumed preferences, he will never willingly stop servicing debt or risk a rollover freeze of short-term debt. In other words, agent 3 is balance-sheet insolvent but not cash-flow insolvent in these states.

To emphasize the interpretation of agent 3 as an arbitrageur, we will from now on assume that $\alpha_3 = \beta_3 = 0$ so that his initial wealth is zero. This in turn implies that the initial endowments of the other agents can be summarized by the pair $(\alpha, \beta) = (\alpha_2, \beta_2)$ that describes the initial portfolio of the constrained agent. In what follows, we assume $\alpha \in [0, 1)$, so that both agents 1 and 2 start with a long position in the stock.

Remark 1. Since $w_3 = 0$, agent 3 can be constructed as a representative arbitrageur that aggregates a finite set of heterogenous arbitrageurs with initial wealth zero, logarithmic preferences and solvency constraint given by $W_{at} \geq -\psi_a S_t$ with $\sum_a \psi_a = \psi$. As a result, a change in the parameter ψ can be interpreted as either a change in credit conditions or as a change in the mass of arbitrageurs present in the economy.

2.5 Definition of equilibrium

The concept of equilibrium that we use is similar to that of equilibrium of plans, prices and expectations introduced by Radner (1972):

Definition 1. *An equilibrium is a pair of security price processes* (S, S_0) *and a set* $\{c_k, (\pi_k; \phi_k)\}_{k=1}^3$ *of consumption plans and trading strategies such that:*

1. *Given* (S, S_0) *the consumption plan* c_k *maximizes* U_k *over the feasible set of agent* k *and is financed by the trading strategy* (π_k, ϕ_k)*.*

2. *Markets clear:* $\phi_1 + \phi_2 + \phi_3 = 0$, $\pi_1 + \pi_2 + \pi_3 = S$ *and* $c_1 + c_2 + c_3 = \delta$*.*

An equilibrium is said to have arbitrage activity if the consumption plan of the arbitrageur is not identically zero.

Since the arbitrageur starts from zero wealth it might be that the set of consumption plans that he can finance is empty. In such cases, his consumption and optimal portfolio are set to zero and the equilibrium only involves the two other agents. To determine conditions under which the arbitrageur participates it is necessary to characterize his feasible set. This is the issue to which we now turn.

2.6 Feasible sets and bubbles

Let (S, S_0) denote the securities prices in a given equilibrium and assume that there are no trivial arbitrage opportunities for otherwise the market could not be in equilibrium. As is well-known (see e.g., Duffie (2001)), this assumption implies that

$$\mu_t = r_t + \sigma_t \theta_t.$$

for some process θ such that $\int_0^t \theta_u^2 du < \infty$ for all $t \geq 0$. This process is referred to as the market price of risk and is uniquely defined on the set where the stock volatility is non zero. Now consider the state price density defined by

$$\xi_t = \frac{1}{S_{0t}} \exp\left(- \int_0^t \theta_u dZ_u - \frac{1}{2} \int_0^t |\theta_u|^2 du \right). \tag{4}$$

The following proposition shows that the ratio $\xi_{t,u} = \xi_u / \xi_t$ can be used as a pricing kernel in order to characterize the feasible sets of agents 1 and 3, and allows to determine the conditions under which the arbitrageur participates in the market.

Proposition 1. *A consumption plan c is feasible for agent $k \in \{1, 3\}$ if and only if*

$$E\left[\int_0^\infty \xi_t c_t dt \right] \leq w_k + 1_{\{k=3\}} \psi(S_0 - F_0), \tag{5}$$

where

$$F_t \equiv E_t \left[\int_t^\infty \xi_{t,u} \delta_u du \right]$$

gives the minimal amount that agent 1 needs to hold at time $t \geq 0$ to replicate the dividends of the stock while maintaining nonnegative wealth. In particular, the feasible set of agent 3 is non empty if and only if $\psi(S_0 - F_0) > 0$.

Following the rational asset pricing bubble literature (see Santos and Woodford (1997) and Loewenstein and Willard (2000) among others) we refer to F_t as the fundamental value of the stock because it is the value attributed to the stock by the unconstrained agent; and to

$$B_t \equiv S_t - F_t = S_t - E_t \left[\int_t^\infty \xi_{t,u} \delta_u du \right]$$

as the bubble on its price. Using this terminology, Proposition 1 shows that the feasible set of the arbitrageur is empty unless two conditions are satisfied: there needs to be a bubble on the stock and the agent must have access to uncollateralized credit in the sense that $\psi > 0$. The intuition behind this result is clear: since the agent has no initial wealth he can only consume strictly positive amounts if there are arbitrages in the market that he is able to exploit, at least to a limited extent.

At first glance, it might seem that a stock bubble should be inconsistent with both optimal choice and the existence of an equilibrium since it implies that two assets with the same cash flows have different prices. The reason why this is not so is that, due to wealth constraints, bubbles only constitute limited arbitrage opportunities. In order to see this, assume the stock has a bubble and consider the textbook arbitrage strategy that sells short $x > 0$ units of the stock, buys the portfolio which replicates the corresponding dividends and invests the remaining strictly positive amount in the riskless asset until some fixed date τ. The wealth process of this strategy is

$$A_{t,\tau}(x) = x(F_t(\tau) - S_t) + xS_{0t}(F_0(\tau) - S_0) = x(B_0(\tau)S_{0t} - B_t(\tau)) \tag{6}$$

where

$$F_t(\tau) \equiv E_t\left[\int_t^\tau \xi_{t,u}\delta_u du + \xi_{t,\tau}S_\tau\right]$$

gives the fundamental value of the stock over the time interval $[t, \tau]$, and

$$B_t(\tau) \equiv S_t - F_t(\tau) \tag{7}$$

denotes the corresponding finite horizon bubble. The portfolio in (6) requires no initial investment and has terminal value $A_{\tau,\tau}(x) = xB_0(\tau)S_{0\tau} > 0$ so it does constitute an arbitrage opportunity in the usual sense. But this arbitrage opportunity is risky because is entails the possibility of interim losses and, therefore, cannot be implemented to an arbitrary scale by the agents in the economy. Indeed, the arbitrageur can only implement this strategy up to size ψ because otherwise the wealth process $A_{t,\tau}(x)$ would not satisfy the solvency constraint (3). Similarly, the unconstrained agent can only implement this strategy if he holds a sufficient amount of collateral in the form of cash or securities. For example, if he holds $x > 0$ units of the stock then he can implement the arbitrage trade only up to size x since the corresponding wealth process $A_{t,\tau}(x) + xS_t$ is nonnegative. These simple arguments show that in the presence of wealth constraints bubble only constitute limited arbitrage opportunities and implies that they are compatible with both individual optimality and the existence of an equilibrium.

The above discussion has focused on the stock but bubbles may be defined on any security, including the money market account. Indeed, over the finite time interval $[0, \tau]$ the money market account can be viewed as a derivative that pays a single lump dividend equal to $S_{0\tau}$ at time τ. The fundamental value of such a security is

$$F_{0t}(\tau) = E_t\left[\xi_{t,\tau} S_{0\tau}\right]$$

whereas its market value is simply S_{0t}, and this naturally leads to defining the finite horizon bubble on the riskless asset as

$$B_{0t}(\tau) \equiv S_{0t} - F_{0t}(\tau) = S_{0t}\left(1 - E_t\left[\xi_{t,\tau}\frac{S_{0\tau}}{S_{0t}}\right]\right). \tag{8}$$

As was the case for stocks, bubbles on the riskless asset are consistent with both optimal choice and the existence of an equilibrium in our economy. In fact, we show below that when constrained agents are present in the economy bubbles on both the stocks and the riskless asset are necessary for markets to clear.

Remark 2. Equation (8) shows that the riskless asset has a bubble over $[0, \tau]$ if and only if the process $M_t \equiv S_{0t}\xi_t$ satisfies $E[M_\tau] < M_0 = 1$. Since the economy is driven by a single source of risk this process is the unique candidate for the density of the risk-neutral probability measure and it follows that the existence of a bubble on the riskless asset is equivalent to the non existence of the risk-neutral probability measure. See Loewenstein and Willard (2000) and Heston, Loewenstein, and Willard (2007).

3 Equilibrium

3.1 Individual optimality

Combining Proposition 1 with well-known results on logarithmic utility maximization leads to the following characterization of optimal policies.

Proposition 2. *Assume that equilibrium prices are such that $B_t \neq 0$. Then the optimal consumption and trading strategies of the three agents are given by*

$$c_{kt} = \rho\left(W_{kt} + 1_{\{k=3\}}\psi B_t\right) \tag{9}$$

and

$$\pi_{1t} = (\theta_t/\sigma_t)W_{1t}, \tag{10}$$

$$\pi_{2t} = (\theta_t/\sigma_t)\kappa_t W_{2t}, \tag{11}$$

$$\pi_{3t} = (\theta_t/\sigma_t)(W_{3t} + \psi B_t) - \psi(\Sigma_t^B/\sigma_t), \tag{12}$$

where

$$\kappa_t = \min\left(1; \frac{(1-\varepsilon)\sigma_\delta}{|\theta_t|}\right), \tag{13}$$

and the process Σ_t^B denotes the diffusion coefficient of the process B_t.

The solution for the unconstrained agent 1 is standard given logarithmic preferences. Indeed, this agent invests in an instantaneously mean-variance efficient portfolio and has a constant marginal propensity to consume equal to his discount rate. The solution for the constrained agent 2 follows from the results of Cvitanić and Karatzas (1992) and shows that the constraint binds when the market price of risk is high. This is intuitive: because agent 2 has logarithmic preferences we know that without the constraint he would invest in proportion to the market of risk and the conclusion follows by noting that the portfolio constraint limits the amount of risk he is allowed to take.

The solution for the arbitrageur is novel to this paper and illustrates how this agent is able to reap arbitrage profits, and thereby consume, despite the fact that he has no initial capital. Specifically, equation (12) shows that the optimal strategy of the arbitrageur consists in shorting ψ units of the stock, buying ψ units of the portfolio that replicates the stock dividends, and then investing the strictly positive proceeds into the same mean-variance efficient portfolio as agent 1. This strategy is only admissible because the arbitrageur is not required to maintain nonnegative wealth and allows him to increase his implicit wealth from W_{3t} to $W_{3t} + \psi B_t = e^{-\rho t}\psi B_0/\xi_t \geq 0$.[8] The optimal consumption in equation (9) then follows by noting that, since he has logarithmic preferences, the arbitrageur should optimally consume a constant fraction of his implicit wealth that is equal to his subjective discount rate.

Remark 3. Interestingly, the optimal policy of the arbitrageur bears a close resemblance

[8]Note that since $W_{3t} + \psi S_t = \psi(F_t + e^{-\rho t}B_0/\xi_t) > 0$ the arbitrageur never exhausts his credit limit. Although of a different nature, this underinvestment result is reminiscent of Liu and Longstaff (2004) who study the portfolio choice problem of an arbitrageur facing margin constraints and an exogenous arbitrage opportunity modeled as a Brownian bridge.

to that of an hypothetical agent with logarithmic utility and no initial wealth who receives labor income at rate $e_t \geq 0$ in a complete market with state price density ξ_t. Indeed, the optimal consumption of such an agent is

$$c_t = \rho(W_t + H_t) = \rho e^{-\rho t}(H_0/\xi_t)$$

where the process

$$H_t = E_t\left[\int_t^\infty \xi_{t,u} e_u du\right]$$

gives the fundamental value of the agent's future income, and an application of Itô's lemma then shows that his optimal trading strategy is

$$\pi_t = (\theta_t/\sigma_t)(W_t + H_t) - (\Sigma_t^H/\sigma_t)$$

where Σ_t^H denotes the diffusion coefficient of the process H_t. This solution is isomorphic to that given in Proposition 2 with one important caveat: instead of arising exogenously from the agent's labor income, the process $H_t = \psi B_t$ in this paper is endogenously generated by the profits that the arbitrageurs are able to reap from the market.

3.2 Equilibrium price system

To characterize the equilibrium we use a representative agent with stochastic weights that allows to easily account for the market clearing conditions despite the imperfect risk sharing induced by the presence of the constrained agent (see Cuoco and He (1994)).[9] The utility function of this representative agent is defined by

$$u\left(c, \gamma, \lambda_t\right) \equiv \max_{c_1+c_2+c_3=c} \left(\log(c_1) + \lambda_t \log(c_2) + \gamma \log(c_3)\right)$$

where $\lambda_t > 0$ is an endogenously determined weighting process that encapsulates the differences across the agents and $\gamma \geq 0$ is a nonnegative constant that determines the relative weight of arbitrageurs in the economy.

[9]This construction is very useful as it reduces the search for an equilibrium to the specification of the weights but one should be cautious with its interpretation because a no-trade equilibrium for the representative agent cannot be decentralized into an equilibrium for our three agents economy in general. The reason for this discrepancy is precisely that the equilibrium prices of our economy can include bubbles whereas those of the representative agent economy cannot.

Relying on the result of Proposition 2, we have that the first order conditions of optimality for agents 1 and 3 are given by

$$e^{-\rho t}\frac{c_{k0}}{c_{kt}} = \xi_t, \qquad k = 1, 3.$$

Comparing these to the first order condition of representative agent's problem shows that the equilibrium state price density and allocation are given by

$$\xi_t = e^{-\rho t}\frac{u_c(\delta_t, \gamma, \lambda_t)}{u_c(\delta_0, \gamma, \lambda_0)} = e^{-\rho t}\frac{\delta_0(1+\gamma+\lambda_t)}{\delta_t(1+\gamma+\lambda_0)}, \tag{14}$$

and

$$\begin{aligned}
c_{2t} &= s_t\delta_t,\\
c_{1t} &= \frac{1}{1+\gamma}(1-s_t)\delta_t,\\
c_{3t} &= \delta_t - c_{1t} - c_{2t} = \frac{\gamma}{1+\gamma}(1-s_t)\delta_t,
\end{aligned}$$

where the process

$$s_t \equiv \frac{c_{2t}}{\delta_t} = \frac{\lambda_t}{1+\gamma+\lambda_t} \in (0,1) \tag{15}$$

represents the consumption share of the constrained agent. In order to determine the dynamics of this process, let us assume that

$$ds_t = m_t dt + n_t dZ_t$$

for some adapted processes m and n. Applying Itô's lemma to the definition of the state price density and comparing the result to (4) shows that the market price of risk and the interest rate are given by

$$\theta_t = \sigma_\delta - \frac{n_t}{1-s_t}, \tag{16}$$

and

$$r_t = \rho + \mu_\delta - \sigma_\delta^2 + \frac{\sigma_\delta n_t - m_t}{1-s_t} - \left(\frac{n_t}{1-s_t}\right)^2. \tag{17}$$

On other hand, the result of Proposition 2 shows that along the optimal path the wealth of agent 2 is $W_{2t} = s_t\delta_t/\rho$. Applying Itô's lemma to this expression, and comparing the result with the dynamic budget constraint (2), shows that the drift and volatility of the consumption share are related by

$$m_t + \frac{n_t^2}{1-s_t} = 0,$$

and that the optimal portfolio of the constrained agent solves

$$\sigma_t\pi_{2t} = W_{2t}\left(\sigma_\delta + \frac{n_t}{s_t}\right).$$

Plugging the expression for the market price of risk into equation (11) and comparing the result with the above expression shows that

$$\left(\sigma_\delta + \frac{n_t}{s_t}\right)\max\left\{1; \frac{|\sigma_\delta(1-s_t) - n_t|}{(1-\varepsilon)(1-s_t)\sigma_\delta}\right\} = \sigma_\delta - \frac{n_t}{1-s_t}.$$

Solving that nonlinear equation gives an explicit expression for the volatility of the consumption share process, and plugging this explicit solution back into equations (16) and (17), delivers the following characterization of equilibrium.

Proposition 3. *In equilibrium, the riskless rate of interest and the market price of risk are explicitly given by*

$$\theta_t = \sigma_\delta\left(1 + \frac{\varepsilon s_t}{1-s_t}\right), \tag{18}$$

$$r_t = \rho + \mu_\delta - \sigma_\delta\theta_t = \rho + \mu_\delta - \sigma_\delta^2\left(1 + \frac{\varepsilon s_t}{1-s_t}\right), \tag{19}$$

and the consumption share of the constrained agent evolves according to

$$ds_t = -s_t\varepsilon\sigma_\delta\left(dZ_t + \frac{s_t}{1-s_t}\varepsilon\sigma_\delta dt\right) \tag{20}$$

with initial condition $s_0 = \rho w_2/\delta_0$.

The above characterization of equilibrium is notable two reasons. First, it follows from (11), (13) and (18) that the equilibrium portfolio of agent 2 satisfies

$$\sigma_t\pi_{2t} = W_{2t}(1-\varepsilon)\sigma_\delta < W_{2t}\theta_t.$$

This shows that the portfolio constraint binds at all times and it follows that agent 2 constantly has a positive demand for the riskless asset. This in turn implies that prices should adjust to entice agents 1 and 3 to borrow and explains why, as shown by (19) and (18), the interest rate decreases and the market price of risk increases compared to an unconstrained economy ($\varepsilon = 0$). Second, (20) shows that the consumption share of the constrained agent is negatively correlated with dividends and therefore tends to decrease (increase) following sequences of positive (negative) cash flow shocks. The intuition for this result is clear: by limiting the amount of risk that agent 2 is allowed to take, the portfolio constraint implies that his consumption is less sensitive to bad shocks but also limits the extent to which it benefits from sequences of good shocks.

To compute the equilibrium price of the stock, we rely on the financial market clearing conditions which require $S_t = \sum_{k=1}^{3} W_{kt}$. Combining this identity with (9), (14) and the clearing of the consumption good market gives

$$S_t = \frac{\delta}{\rho} - \psi(S_t - F_t) \tag{21}$$

where

$$F_t = E_t\left[\int_t^\infty \xi_{t,u}\delta_u du\right] = \delta_t(1 - s_t)E_t\left[\int_t^\infty e^{-\rho(u-t)}\frac{du}{1 - s_u}\right]$$

gives the fundamental value of the stock. Setting $\nu = \psi/(1+\psi)$ and solving for the stock price we finally arrive at

$$S_t = \nu F_t + (1 - \nu)\frac{\delta_t}{\rho}.$$

This expression shows that when arbitrageurs are absent from the economy ($\nu = 0$) the equilibrium stock price is given by $P_t \equiv \delta_t/\rho$ which is standard in economies with logarithmic preferences. On the other hand, when arbitrageurs are present the equilibrium price includes an additional component that is equal to $-\psi B_t$. This term is negative and it is nonzero if and only if the stock price includes a bubble component in which case it reflects the negative impact of arbitrage activity on the value of the collateral services provided by the stock.

To complete the description of the equilibrium, it remains to determine whether the stock price includes a bubble component or not. Using the definition of the bubble together with the above expression, and the relation between the consumption share and the weighting process in (15), we obtain

$$B_t = (1-\nu)(P_t - F_t) = (1-\nu)\delta_t E_t \left[\int_t^\infty e^{-\rho(u-t)} \left(\frac{\lambda_t - \lambda_u}{1+\gamma+\lambda_t} \right) du \right] \tag{22}$$

and it follows that the stock price is bubble free if and only if the weighting process is a martingale. Applying Itô's lemma to the weighting process gives

$$d\lambda_t = (1+\gamma) d\left(\frac{s_t}{1-s_t} \right) = -\lambda_t (1+\gamma+\lambda_t) \frac{\varepsilon\sigma_\delta}{1+\gamma} dZ_t,$$

so that the weighting process is a local martingale. However, the following proposition shows that this local martingale fails to be a true martingale, and thereby proves that any equilibrium has arbitrage activity.

Proposition 4. *The weighting process is a strict local martingale. In particular, the stock price includes a strictly positive bubble component in any equilibrium.*

The following theorem relies on the properties of the weighting process to derive closed form expressions for the stock price and the bubble on the stock, as well as parametric conditions for existence and uniqueness of equilibrium.

Theorem 1. *Let*

$$g(s) \equiv \beta - P_0(s - \alpha(1 - \nu s^\eta)), \tag{23}$$

with the constant

$$\eta \equiv \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\rho}{(\varepsilon\sigma_\delta)^2}}$$

and assume that the parameters of the model are such that

$$g(1) < 0 < g(0) = \beta + \alpha P_0 < P_0 \left[1 - \nu(1-\alpha)(\beta/((1-\alpha)P_0)^\eta \right]. \tag{24}$$

The price of the stock and its bubble component are given by

$$S_t = (1 - \nu s_t^\eta) P_t, \tag{25}$$

$$B_t = (1-\nu) s_t^\eta P_t, \tag{26}$$

and the consumption share process s_t evolves according to (20) *with initial condition* $s_0 = s^*$ *where* $s^* \in (0,1)$ *is the unique solution to* $g(s^*) = 0$.

As shown by (25) the equilibrium stock price is lower than the price $P_t = \delta_t/\rho$ that would prevail in an economy with no bubbles, or in an economy where only agents 1 and 2 trade. We will be back to this issue in the analysis below, but notice for the moment that, unless $\alpha = 0$ in which case agent 1 holds the whole supply of the stock, the fundamental value of the stock

$$F_t = S_t - B_t = (1 - s_t^\eta)\, P_t$$

is not independent from the bubble since the path of the consumption share process depends on the initial distribution of wealth in the economy which in turn depends on the equilibrium stock price, and therefore on the size of the bubble.[10]

The next proposition shows that, in addition to a bubble in the price of the stock, the equilibrium price system also includes a non trivial bubble in the price of the riskless asset over any finite horizon.

Proposition 5. *Assume that the conditions of Theorem 1 hold. Then the equilibrium price of the riskless asset is*

$$S_{0t} = e^{qt} \frac{P_t}{P_0} \left(\frac{s_t}{s_0} \right)^{1/\varepsilon} \tag{27}$$

with the constant defined by

$$q = \rho - \frac{1}{2}(1-\varepsilon)\sigma_\delta^2.$$

Over a time interval of length $\tau > 0$ the equilibrium prices of the stock and the riskless asset include bubble components that are given by

$$B_t(t+\tau) = H\left(\tau, s_t; 2\eta - 1\right) B_t, \tag{28}$$

$$B_{0t}(t+\tau) = H\left(\tau, s_t; 2/\varepsilon - 1\right) S_{0t}, \tag{29}$$

[10]See Weil (1990) for another example of an economy in which the fundamental value of the asset depends on the size of the bubble component and thereby generates an equilibrium price that is lower than in the absence of the bubble.

where

$$H(\tau, s; a) \equiv N(d_+(\tau, s; a)) + s^{-a} N(d_-(\tau, s; a)), \tag{30}$$
$$d_\pm(\tau, s; a) \equiv \frac{1}{\varepsilon\sigma_\delta\sqrt{\tau}} \log(s) \pm \frac{a}{2}\varepsilon\sigma_\delta\sqrt{\tau},$$

and the function $N(\cdot)$ denotes the cumulative distribution function of a standard normal random variable.

We stress the fact that the emergence of the bubble component in the price process responds to a clear equilibrium mechanism: shifts in the interest rate and market price of risk go on the right direction but are not sufficient to reach an equilibrium, so bubbles arise to incite agents 1 and 3 to hold positions that are compatible with market clearing.

4 Analysis

In this section, we show that the mispriced assets are brought closer to their fundamental values as the availability of credit grows, and that the arbitrageur's trading activity generates excess volatility and the leverage effect. We also explore welfare implications.

4.1 Portfolio strategies and bubble sizes

The following proposition provides an explicit representation of equilibrium portfolio strategies.

Proposition 6. *Portfolio positions (and their respective signs) are given by*

$$(\underset{+}{\pi_{1t}}, \underset{-}{\phi_{1t}}) = \left(\frac{1-(1-\varepsilon)s_t}{(1+\gamma)v(s_t)}, -\frac{1-(1-\varepsilon)s_t-(1-s_t)v(s_t)}{(1+\gamma)v(s_t)} \right) P_t,$$
$$(\underset{+}{\pi_{2t}}, \underset{+}{\phi_{2t}}) = \left(\frac{(1-\varepsilon)s_t}{v(s_t)}, \frac{v(s_t)s_t-(1-\varepsilon)s_t}{v(s_t)} \right) P_t,$$
$$(\underset{+/-}{\pi_{3t}}, \underset{-}{\phi_{3t}}) = \left(\frac{h(s_t)+\gamma(1-s_t)v(s_t)}{(1+\gamma)v(s_t)} - \nu s_t^\eta, -\frac{h(s_t)}{(1+\gamma)v(s_t)} \right) P_t,$$

where

$$v(s) = 1 + \frac{\nu\eta\varepsilon s^\eta}{1-\nu s^\eta},$$
$$h(s) = (1+\gamma s)(v(s)-1) + \varepsilon\gamma s.$$

Since the portfolio constraint binds at all times, agent 2 must hold a positive position on the riskless asset that is offset by borrowing positions held by agents 1 and 3. It is notable that agent 1 optimally chooses to hold a levered position on the stock even though the stock also contains a bubble component. To understand this choice, notice that he exploits the bubble on the riskless asset because it requires less collateral per unit of initial profit.

Proposition 7. *Let $\tau > t$. Bubbles are such that*

$$\frac{B_{0t}(\tau)}{S_{0t}} - \frac{B_t(\tau)}{S_t} \geq 0,$$

The following proposition confirms the above intuition, as it shows that the wealth of unconstrained agent can be decomposed into a portfolio that uses the stock as collateral and shorts the bubble on the riskless asset.

Proposition 8. *The wealth of the unconstrained agent expressed as positions in the stock and in the riskless asset's bubble over horizon $(t, \tau]$ is given by*

$$W_{1t} = \phi_{1t}^{S}(\tau) + \phi_{1t}^{B_0}(\tau), \tag{31}$$

where

$$\underset{+}{\phi_{1t}^{S}}(\tau) = \frac{(1 - \Sigma_t^0(\tau))(1 - s_t) + \varepsilon s_t}{(1+\gamma)(v(s_t) - \Sigma_t^0(\tau))} P_t, \tag{32}$$

$$\underset{-}{\phi_{1t}^{B_0}}(\tau) = -\frac{1 - (1-\varepsilon)s_t - (1 - s_t)v(s_t)}{(1+\gamma)(v(s_t) - \Sigma_t^0(\tau))} P_t \tag{33}$$

and the process

$$\Sigma_t^0(\tau) = -\frac{\varepsilon s_t \partial H(\tau - t, s_t; 2/\varepsilon - 1)/\partial s}{H(\tau - t, s_t; 2/\varepsilon - 1)} < 0,$$

corresponds to the diffusion coefficient of the process $(1/\sigma_\delta) \log B_{0t}(\tau)$.

This strategy converges to the standard representation in (1) as $\tau \to \infty$.[11] Figure 1 shows comparative statics for changes in the severity of the constraint (ε) and the size of the credit facility (ψ).

[11]The fundamental value of the riskless asset converges to zero as $\tau \to \infty$, which implies that the money market account in this economy is akin to fiat money, $\lim_{\tau \uparrow \infty} B_{0t}(\tau) = S_{0t}$.

Insert Figure 1 here

The arbitrageur's strategy, on the other hand, mimics agent 1's strategy while financing it by shorting the bubble on the stock,

$$W_{3t} = \gamma W_{1t} - \psi B_t.$$

Access to the credit facility allows the arbitrageur to be more aggressive than agent 1. In particular, the arbitrageur may be even able to sustain short positions on both securities as seen in Proposition 6. As a result, the arbitrageur holds a much larger short position in the riskless asset bubble relative to his position in the stock, which when aggregated, amount to a lower stock level required to clear the stock market. This mechanism is behind the reduction in the value of the stock's collateral services and consequently the decrease in the relative size of the bubble, as illustrated in the left panel of Figure 2.

Insert Figure 2 here

One simple way to see the equilibrium effect described above is by noting that the stock and arbitrage profits finance aggregate consumption, $S_0 + \psi B_0 = \delta_0/\rho$. Since $\frac{\partial}{\partial \psi}\psi B_0 > 0$, increasing ψ reduces necessarily the stock level since aggregate consumption is invariant to the presence of the credit facility.

Level effects are not only restricted to the bubble on the stock. The bubble on the riskless asset also decreases with a credit improvement, as seen in the right panel of Figure 2. Note however that the bubble on the riskless asset survives as $\psi \to \infty$.[12]

4.2 Volatility and the leverage effect

In a noteworthy departure from previous models with frictions and homogenous agents with logarithmic utility,[13] the price dividend ratio in (25) is decreasing in the consumption share of the constrained agent as a result of the arbitrageur's trading activity. Since the consumption share of the constrained agent is negatively correlated with dividends, the stock volatility is higher than the volatility of dividends in all states.

[12]The limiting result is in line with Loewenstein and Willard (2000) who show that bubbles can generally exist on securities prices in zero net supply.

[13]See for example Detemple and Murthy (1997), Basak and Cuoco (1998) and Basak and Croitoru (2000). In Xiong (2001), 'convergence traders' have logarithmic utility and reduce volatility on average, but they can also increase volatility in some circumstances in the presence of noise traders.

Corollary 1. *The volatility of the stock is given by*

$$\sigma_t = v(s_t)\sigma_\delta = \sigma_\delta + \frac{\sigma_\delta \nu \eta \varepsilon s^\eta}{1 - \nu s^\eta}. \tag{34}$$

The second term in (34) corresponds to what is commonly referred to as the excess volatility component. This term is positive and increasing in the consumption share, an observation which coupled with the fact that the price dividend ratio is decreasing in the consumption share, implies that the model with arbitrage activity generates the so-called leverage effect, i.e., the well-established stylized fact according to which volatility increases when the stock price falls.[14] Note also that volatility is increasing in the size of the credit facility

$$\frac{\partial \sigma_0}{\partial \nu} = \frac{(1-\nu)^2 s_0^{1+\eta} \sigma_\delta \varepsilon \eta}{(1-\nu s_0^\eta)^2 (s_0 + \nu \alpha \eta s_0^\eta)} > 0. \tag{35}$$

4.3 Welfare

The impact of an improvement on credit conditions on the agents' welfare is summarized in the following proposition.

Proposition 9. *The arbitrageur benefits from an improvement of the credit conditions ($\psi \uparrow$) at the expense of possibly both agents 1 and 2.*

The expected utility of the arbitrageur

$$\begin{aligned} U_3(\psi) &\equiv E\left[\int_0^\infty e^{-\rho t} \log(c_{3t}) dt\right] \\ &= U_0 + \rho^{-1} \log\left(\frac{\gamma}{1+\gamma}\right) + E\left[\int_0^\infty e^{-\rho t} \log(1 - s_t) dt\right], \end{aligned} \tag{36}$$

where $U_0 \equiv E\left[\int_0^\infty e^{-\rho t} \log(\delta_t) dt\right]$, reveals that this result follows from the fact that changes in ψ impact the consumption sharing rules (γ, s) through price shifts. In particular, the second term in (36) is increasing in ψ, since the relative weight of the arbitrageur, $\gamma = \psi B_0 / w_1$, increases as the trading activity of the arbitrageur reduces the stock price. The third term in (36) is also increasing in ψ because the starting point of the consumption share is decreasing in the credit conditions and the path of s depends monotonically on its starting point. The latter means that agent 2's welfare is decreasing in ψ.

[14] See Schwert (1989) and Mele (2007) for evidence on the asymmetric nature of volatility and the cyclical behavior of stock prices.

A similar analysis shows that the welfare of agent 1, represented by

$$\begin{aligned} U_1(\psi) &\equiv E\left[\int_0^\infty e^{-\rho t}\log(c_{1t})dt\right] \\ &= U_0 - \rho^{-1}\log(1+\gamma) + E\left[\int_0^\infty e^{-\rho t}\log(1-s_t)dt\right], \end{aligned} \tag{37}$$

is negatively affected by an improvement of the credit conditions through two channels. First, the second term is decreasing in ψ, since better credit conditions reduce his initial wealth. Second, credit improvements also increase the stock's volatility as shown in (35). The latter effect is detrimental for agent 1 because the stock's 'collateral quality' worsens from the unconstrained agent's standpoint. Indeed, the lower panel in Figure 1 shows that the unconstrained agent drastically reduces his arbitrage position per unit of collateral as $\psi \uparrow$.[15]

The negative impact caused by an improvement in credit conditions may be compensated by a decrease in the share of consumption of the constrained agent, which increases the third term in (37). Note however that if the initial endowment of agent 2 does not depend on the stock ($\alpha = 0$), credit conditions do not have welfare implications for the constrained agent, as the process s is independent of ψ. In this case, an increase in the availability of credit necessarily implies a welfare loss for agent 1, as it will impact the stock price but not the interest rate.

5 Extensions

In this section, we show that similar asset pricing implications are obtained for an alternative credit facility. However, unlike the previous example, equilibrium may fail to exist if the credit facility is too big. We also analyze the implications of a sudden reversal in credit conditions for arbitrageurs.

5.1 Alternative credit facility

Assume that the arbitrageur faces a wealth constraint given by

$$W_{3t} \geq -\ell S_{0t} \tag{38}$$

[15]This effect could also be interpreted as if an increase in the mass of arbitrageurs effectively crowds the unconstrained agent out of the arbitrage business.

for some constant $\ell > 0$. This bound depends on the evolution of the short rate only[16] and requires that temporary losses be bounded by ℓ in discounted terms. Feasible plans satisfy the static budget constraint

$$E\left[\int_0^\infty \xi_t c_{3t} dt\right] \leq \ell,$$

where the term on the right hand side provides the present value of the profits that arbitrageurs are able to reap from the market.

Proceeding as in the baseline example, the candidate stock price now depends linearly on the price of the riskless asset,

$$\begin{aligned} S_t &= F_t + \delta_t E_t \left[\int_t^\infty e^{-\rho(u-t)} \frac{\lambda_t - \lambda_u}{1+\gamma+\lambda_t} du\right] - \ell S_{0t} \\ &= \left(1 - e^{qt} \frac{\ell}{P_0} \left(\frac{s_t}{s_0}\right)^{1/\varepsilon}\right) P_t, \end{aligned} \tag{39}$$

where s evolves according to (20) with starting point given by

$$s_0 = s^* \equiv (\alpha(P_0 - \ell) + \beta)/P_0.$$

The next proposition shows that is possible to construct a unique equilibrium by restricting not only the initial endowments as in (24), but also the size of arbitrage strategies along the equilibrium path.

Proposition 10. *The equilibrium exists if and only if*

$$q = \rho - \frac{1}{2}(1-\varepsilon)\sigma_\delta^2 \leq 0, \tag{40}$$

and

$$P_0(\ell/P_0)^\varepsilon < \beta + \alpha(P_0 - \ell) < P_0 - \ell. \tag{41}$$

(40) *and* (41) *guarantee that* $s^* \in (0,1)$ *while attributing strictly positive wealth to agent 1 and such that the stock price is nonnegative at all times.*

Notably, the equilibrium fails to exist when the size of the credit facility is above a threshold, $\ell \geq P_0$. The intuition is that the size of the arbitrage positions might be too big to be supported by a viable price system. In other words, (39) would be strictly

[16] See Loewenstein and Willard (2000) for a similar wealth constraint.

negative with strictly positive probability.

On the other hand, the equilibrium implications in this economy match our previous results in various dimensions. First, the bubble on the stock[17]

$$B_t = S_t - F_t = \left(s_t^\eta - e^{qt} \frac{\ell}{P_0} \left(\frac{s_t}{s_0} \right)^{1/\varepsilon} \right) P_t$$

is decreasing in the size of the credit facility. The bubble on the riskless asset over a time interval of length τ is given by (29), and as in our baseline example, it decreases with credit improvements, since $s_0'(\ell) = -\alpha/P_0$ and the path of s depends monotonically on its starting point.

Second, the arbitrage activity induced by the credit facility in (38) also generates the leverage effect, since the stock volatility is given by

$$\sigma_t = \left(1 + \frac{\ell S_{0t}}{S_t} \right) \sigma_\delta$$

and is increasing in the consumption share. Finally, volatility is increasing in the size of the credit facility,

$$\frac{\partial \sigma_0}{\partial \ell} = P_0/(P_0 - \ell)^2 \sigma_\delta > 0.$$

5.2 Liquidity shocks

We sketch the impact of an exogenous liquidity shock that gives rise to a sudden and unanticipated change in the funding conditions ψ. In particular, we assume that ψ may drop to zero in states in which the arbitrageur has negative net worth.[18] Upon this sudden change in credit conditions, the stock price adjusts to a new equilibrium with $\psi' = 0$, which, as seen in (25), implies $S' \geq S$.

This liquidity shock may impact agent 2 negatively when the collateral position of the arbitrageur, now valued at $\frac{\pi_3}{S} S'$, does not cover for the arbitrageur's short position $|\phi_3|$.

[17] The inequality in (40) ensures that $B_t \geq 0$.

[18] In this brief exercise, we attempt to capture the fact that financial crises are often preceded by periods of credit expansion during which agents become increasingly vulnerable to a reversal in economic conditions. The crisis literature often uses exogenous liquidity shocks as a modeling device. See Krishnamurthy (2010) for a recent example.

This corresponds to states where the arbitrageur is rendered balance-sheet insolvent

$$\frac{\pi_3}{S} S' \leq |\phi_3|,$$

with no possibility of repayment, since he cannot generate arbitrage profits due to the tightened credit conditions. Interestingly, when $\pi_3 > 0$, agent 2 will be indifferent between forgoing a fraction

$$\varphi \equiv \max\left(0; 1 - (\pi_3 S'/(|\phi_3|S))^+\right) \in [0, 1]$$

of the arbitrageur's debt position, or seizing the arbitrageur's stock holdings, as seen in Table 1.

Agent - State	$\frac{\pi_3}{S} S' \leq 0$	$0 \leq \frac{\pi_3}{S} S' \leq \lvert\phi_3\rvert$
2	$W_2' = \frac{\pi_2}{S} S' + \lvert\phi_1\rvert$	$W_2' = \frac{\pi_2}{S} S' + \lvert\phi_1\rvert + (1 - \varphi)\lvert\phi_3\rvert$
3	$W_3 = \frac{\pi_3}{S} S' + \phi_3 < 0$	$W_3' = \frac{\pi_3}{S} S' + (1 - \varphi)\phi_3 = 0$

Table 1: Portfolio allocations after the liquidity shock $\psi \to 0$

The key feature in states where $0 \leq \frac{\pi_3}{S} S' \leq |\phi_3|$ is that it may imply a redistribution from constrained agents (lenders) to arbitrageurs. Unconstrained agents do not have an incentive to intervene. To see this, note that the liquidity shock makes agent 1 better off, since agent 1 now holds a higher valued position on the stock, $\frac{\pi_1}{S} S' > \frac{\pi_1}{S} S$, and his short position on the money market account remains the same (due to its short term nature).

The analysis above may change significantly if the liquidity shock is coupled with an unanticipated dividend shock such as a downward jump on dividends. In this case, the arbitrageur is hit from two sources: the stock price decreases at the same time that the credit conditions worsens. Notably, this type of liquidity shock may also impact negatively the unconstrained agent.

6 Concluding remarks

This paper studies a pure exchange economy populated by three types of agents: constrained agents who are subject to portfolio constraints, unconstrained agents who are only subject to a standard nonnegativity constraint on wealth, and arbitrageurs who, in addition to being unconstrained, may incur transitory losses that are bounded by a state-dependent credit limit.

We show that this uncollateralized credit facility is valuable when there are asset pricing bubbles and that the mispricing in securities decreases with improved credit conditions, to the extent that the bubble on the stock vanishes in the limit of infinite credit. The equilibrium with risky arbitrage activity is characterized by excess volatility and the leverage effect, as well as a countercyclical market price of risk and low interest rates.

We find that constrained and unconstrained agents may be made worse off as the arbitrageur's trading activity impact prices in a way that may be unfavorable to them.

The model has some of the elements one would require to analyze the implications of a sudden reversal in credit conditions and the possibility of bailouts: heterogeneous access to credit and excess leverage. We sketch some of the implications of unanticipated shocks in credit.

A Proofs

Proof of Proposition 1. The static budget constraint in (5) for agent 1 is a well known result (see e.g., Duffie (2001), Chapter 9.E.). For an arbitrary consumption and investment plan, the deflated wealth process of the arbitrageur is

$$\xi_t W_{3t} + \int_0^t \xi_u c_{3u} du = \int_0^t \xi_s \left(\pi_{3u}\sigma_u - W_{3u}\theta_u\right) dZ_u.$$

The deflated stock and riskless asset price processes satisfy

$$\xi_t S_t + \int_0^t \xi_u \delta_u du = S_0 + \int_0^t \xi_u \left(S_u\sigma_u - S_u\theta_u\right) dZ_u,$$
$$\xi_t S_{0t} = 1 - \int_0^t \xi_u S_{0s}\theta_u dZ_u.$$

Let N_t be defined by

$$\begin{aligned} N_t &= \xi_t W_{3t} + \psi\xi_t S_t + \ell\xi_t S_{0t} + \int_0^t \xi_u (c_{3u} + \psi\delta_u) du \\ &= \psi S_0 + \ell + \int_0^t \xi_u \left((\pi_{3u} + \psi S_u)\sigma_u - (\psi S_u + W_{3u} + \ell S_{0u})\theta_u\right) dZ_u \end{aligned} \tag{A.1}$$

with $\psi \geq 0$ and $\ell \geq 0$. N_t is a nonnegative local martingale for positive consumption plans, and hence a supermartingale, since the price system (S_{0t}, S_t) is nonnegative. This implies that

$$E\left[\int_0^T \xi_t (c_{3t} + \psi\delta_t) dt + \ell\xi_T S_{0T}\right] \leq \psi S_0 + \ell.$$

The static budget constraint in (5) follows by setting $\ell = 0$ and letting $T \to \infty$. ■

Proof of Proposition 2. The optimal policies of the unconstrained agent in (9) and (10) are a well known result (see e.g., Duffie (2001), Chapter 9.E and Karatzas and Shreve (1998), p.32). The optimal policy of the constrained agent follows from Cvitanić and Karatzas (1992). Agent 2 faces an implicit state price representation given by

$$\xi_{2t} = e^{-\int_0^t \left(r_u + \beta_u + \frac{1}{2}\theta_{2u}^2\right) ds - \int_0^t \theta_{2u} dZ_u},$$

where $\theta_{2t} = \theta_t + \sigma_t^{-1}\omega_t$. $\beta_t(\omega)$ is the support function of the set $-\mathcal{C}_t$. Optimality conditions imply that ω is defined by the relation

$$\omega_t = \arg\min_{\omega \in \mathcal{B}_t} \left\{ \frac{1}{2} \left(\theta_t + \sigma_t^{-1} \omega \right)^2 + \beta_t (\omega) \right\}.$$

where $\mathcal{B}$ is the set of points where the support function is finite. Using the definition of the support function and Fenchel's duality theorem (see Rockafellar (1996), Theorem 31.1), the problem above is transformed into a mean-variance program given by

$$\sup_{\hat{\pi} \in \mathcal{C}} \left\{ \hat{\pi} \sigma_t \theta_t - \frac{1}{2} \left(\sigma_t \hat{\pi} \right)^2 \right\}.$$

Since $\mathcal{C}_t$ is a closed convex subset of $\mathbb{R}$, the problem admits a unique solution given by

$$\sigma_t \hat{\pi}_t = \kappa_t \theta_t, \qquad \kappa_t = \min \left(1; \frac{(1-\varepsilon)\sigma_\delta}{|\theta_t|} \right).$$

where $\hat{\pi} = \pi / W$. The optimal policy of the arbitrageur is derived as follows. The consumption process $c_{3t} = (e^{\rho t} y_3 \xi_t)^{-1}$ is optimal if and only if there exists a constant $y_3 > 0$ that satisfies

$$E \left[\int_0^\infty \xi_t (e^{\rho t} y_3 \xi_t)^{-1} dt \right] = \psi \left(S_0 - F_0 \right).$$

The wealth process implied by (9) follows from the fact that N_t in (A.1) evaluated at the optimal is a true martingale,

$$\xi_t W_{3t} + \psi \xi_t S_t + \int_0^t \xi_u (c_{3u} + \psi \delta_u) du = E_t \left[\int_0^\infty \xi_u (c_{3u} + \psi \delta_u) du \right],$$

so that

$$W_{3t} = \frac{c_{3t}}{\rho} - \psi B_t. \tag{A.2}$$

Applying Itô's lemma to (A.2) and matching terms with the process in (2) gives (12). ■

Proof of Proposition 3. The marginal utility of the representative agent is identified from the clearing condition

$$\frac{1}{u_c(\delta_t, \lambda_t)} + \frac{\gamma}{u_c(\delta_t, \lambda_t)} + \frac{\lambda_t}{u_c(\delta_t, \lambda_t)} = \delta_t,$$

so that $u_c(\delta_t, \lambda_t) = \frac{1+\gamma+\lambda_t}{\delta_t}$. The constants (y_3, γ, λ_0) are given by

$$y_3 = \frac{1}{c_{30}} = \frac{1}{\rho\psi B_0}, \quad \gamma = \frac{c_{30}}{c_{10}} = \frac{\psi B_0}{w_1}, \quad \lambda_0 = \frac{c_{20}}{c_{10}} = \frac{w_2}{w_1}.$$

The procedure to compute the interest rate and the market price of risk in (18) and (19) is described in the main text. ■

We make use of the following (adapted) results from Hugonnier (2012).

Lemma A.1. *Let $\tau \geq t$, then*

$$\begin{aligned} q_t(\tau) \equiv& \rho s_t E_t \left[\int_t^\tau e^{-\rho(u-t)}(1 - \lambda_u/\lambda_t)du \right] \\ =& s_t^\eta H(\tau - t, s_t; 2\eta - 1) - e^{-\rho(\tau-t)} s_t H(\tau - t, s_t, 1). \end{aligned} \tag{A.3}$$

where the function H is defined in (30).

Proof. See Lemma A.3 in Hugonnier (2012). ■

Lemma A.2. *Take the process*

$$X_t \equiv \frac{s_t}{1 - s_t} = \frac{\lambda_t}{1 + \gamma} \tag{A.4}$$

whose dynamics follow

$$dX_t = -X_t (1 + X_t) \varepsilon \sigma_\delta dZ_t.$$

Let $a \in \mathbb{R}$ be an arbitrary constant. Then the expectation function of the nonnegative local martingale

$$Y_t(a) = 1 - \int_0^t Y_u(a)(a + X_u)\varepsilon\sigma_\delta dZ_u \tag{A.5}$$

is explicitly given by

$$E_t [Y_\tau(a)] = Y_t(a) (1 - H(\tau - t, s_t; 2a - 1)) \tag{A.6}$$

where the function H is defined in (30). In particular, the unique solution to equation (A.5) is a strictly positive local martingale but it is not a true martingale.

Proof. See Lemma A.5 in Hugonnier (2012). ■

Proof of Proposition 4. Applying Itô's lemma to the weighting process λ gives

$$\frac{d\lambda_t}{\lambda_t} = [\beta_t(\nu) + \theta_{2t}(\theta_{2t} - \theta_t)]\, dt + (\theta_{2t} - \theta_t)\, dZ_t. \tag{A.7}$$

From the proof of Proposition 2,

$$\theta_{2t} = \kappa_t \theta_t, \quad \omega_t = -(1-\kappa_t)\sigma_t \theta_t.$$

Replacing these results in the drift of equation (A.7) gives

$$\beta_t(\nu) + \theta_{2t}(\theta_{2t} - \theta_t) = \kappa_t(1-\kappa_t)(\sigma_t^{-1}\theta_t)\sigma_t\theta_t - \kappa_t(1-\kappa_t)\theta_t\theta_t = 0,$$

so that its dynamics evolve according to

$$\begin{aligned} d\lambda_t &= -\lambda_t(1+\gamma+\lambda_t)\frac{\varepsilon\sigma_\delta}{1+\gamma} dZ_t \\ &= -\lambda_t(1+X_t)\varepsilon\sigma_\delta dZ_t. \end{aligned}$$

The result follows from Lemma A.2 by setting $\lambda_t = Y_t(1)$. ■

Proof of Theorem 1. From (22), the bubble on the stock is given by

$$\begin{aligned} B_t =& (1-\nu)\delta_t E_t\left[\int_t^\infty e^{-\rho(u-t)}\left(\frac{\lambda_t - \lambda_u}{1+\gamma+\lambda_t}\right) du\right] \\ =& (1-\nu)P_t \frac{\lambda_t}{1+\gamma+\lambda_t}\rho E_t\left[\int_t^\infty e^{-\rho(u-t)}(1-\lambda_u/\lambda_t)\, du\right] \\ =& (1-\nu)P_t \lim_{\tau\to\infty} q_t(\tau), \end{aligned}$$

where $q_t(\tau)$ is given in Lemma A.1. The result in (26) follows from taking the corresponding limit in (A.3). The stock price in (25) follows from using the above result in (21) and simplifying terms. Existence follows from ensuring that the constants (γ, λ_0) are strictly positive. This is equivalent to finding conditions such that $s_0 \in (0,1)$ and

$$w_1(s_0) = (1-\alpha)P_0(1-\nu s_0^\eta) - \beta = P_0(1-\nu s_0^\eta - s_0) > 0, \tag{A.8}$$

since $\gamma = \frac{\psi B_0}{w_1}$ and $\lambda_0 = \frac{w_2}{w_1}$. The function $g(s)$ in (23) corresponds to the initial wealth of agent 2 minus his wealth expressed using the consumption sharing rule. Note that $g(s_0)$

is continuous and decreasing,

$$g'(s) = -P_0(\alpha\eta\nu s^{\eta-1} + 1) < 0. \tag{A.9}$$

The following conditions ensure there is a root $s^* \in (0, 1)$,

$$g(0) = \beta + \alpha P_0 > 0, \qquad g(1) = \beta + P_0\left(\alpha(1-\nu) - 1\right) < 0. \tag{A.10}$$

On the other hand, for $w_1(s^*) > 0$

$$1 - \nu(s^*)^\eta > \beta/((1-\alpha)P_0), \tag{A.11}$$

$$1 - \nu(s^*)^\eta - s^* > 0. \tag{A.12}$$

Conditions in (24) follow from (A.10), (A.11) and (A.12). Uniqueness follows from (A.9). ■

Proof of Proposition 5. We use the equilibrium interest rate and the dynamics of the consumption share of the constrained agent to compute the price of the riskless asset in (27). From (7), the definition of $F_t(\tau)$ and the law of iterated expectations,

$$\begin{aligned}
B_t(\tau) =& S_t - E_t\left[\int_t^\tau \xi_{t,u}\delta_u du + \xi_{t,\tau}S_\tau\right] \\
=& S_t - E_t\left[\int_t^\infty \xi_{t,u}\delta_u du - \int_\tau^\infty \xi_{t,u}\delta_u du + \xi_{t,\tau}S_\tau\right] \\
=& B_t - E_t\left[\xi_{t,\tau}\left(S_\tau - E_\tau \int_\tau^\infty \xi_{\tau,u}\delta_u du\right)\right] \\
=& B_t - E_t\left[\xi_{t,\tau}B_\tau\right].
\end{aligned} \tag{A.13}$$

In order to compute the second term in (A.13), we use the following results:

$$E_t\left[\int_t^\infty e^{-\rho u}\lambda_u du\right] = \frac{e^{-\rho t}}{\rho}(\lambda_t - (1+\gamma+\lambda_t)s_t^\eta), \tag{A.14}$$

$$E_t\left[\int_t^\tau e^{-\rho u}\lambda_u du\right] = \frac{e^{-\rho t}}{\rho}\lambda_t\left(1 - e^{-\rho(\tau-t)}\right) - \frac{e^{-\rho t}}{\rho}\lambda_t\frac{q_t(\tau)}{s_t}. \tag{A.15}$$

(A.14) follows from the definition of $q_t(\cdot)$ and the limit in (A.3). (A.15) follows from the definition of $q_t(\cdot)$. Using (14), (26), (A.14) (A.15) and the fact that from (A.6) we have

$E_t[\lambda_\tau]$ in closed form by setting $\lambda_t = Y_t(1)$, gives

$$\begin{aligned} E_t\left[\xi_\tau B_\tau\right] =& c_{10}(1-\nu)E_t\left[\frac{e^{-\rho\tau}}{\rho}(1+\gamma+\lambda_\tau)s_\tau^\eta\right] \\ =& c_{10}(1-\nu)E_t\left[\frac{e^{-\rho\tau}}{\rho}\lambda_\tau - \int_\tau^\infty e^{-\rho u}\lambda_u du\right] \\ =& c_{10}(1-\nu)E_t\left[\frac{e^{-\rho\tau}}{\rho}\lambda_\tau - \int_t^\infty e^{-\rho u}\lambda_u du + \int_t^\tau e^{-\rho u}\lambda_u du\right] \\ =& c_{10}(1-\nu)\frac{e^{-\rho t}}{\rho}(1+\gamma+\lambda_t)s_t^\eta\left(1 - H(\tau-t, s_t; 2\eta-1)\right). \end{aligned}$$

Combining the above result with (14) in (A.13) gives (28). The process $M_t = S_{0t}\xi_t$ evolves according to

$$dM_t = -M_t\theta_t dZ_t = -M_t\left(1/\varepsilon + X_t\right)\varepsilon\sigma_\delta dZ_t,$$

The bubble on the riskless asset in (29) is given by

$$\begin{aligned} B_{0t}(\tau) &= S_{0t}\left(1 - E_t\left[M_\tau/M_t\right]\right) \\ &= S_{0t}H(\tau-t, s_t; 2/\varepsilon - 1) \end{aligned} \tag{A.16}$$

where (A.16) follows from an application of Lemma A.2 by setting $M_t = Y_t(1/\varepsilon)$. ■

Proof of Proposition 6. The result follows from using equilibrium quantities from Proposition 3 and Theorem 1 in Proposition 2. The sign of ϕ_1 follows from noting that

$$\begin{aligned} \text{sign}\left[\phi_1\right] &= -\text{sign}\left[1-(1-\varepsilon)s-(1-s)v(s)\right] \\ &= -\text{sign}\left[1-\nu s^{\eta-1}((1-s)\eta+s)\right] \end{aligned}$$

The function $s \in (0,1) \to h(s) = 1-\nu s^{\eta-1}((1-s)\eta+s)$ is strictly decreasing with range $(1, 1-\nu)$. ■

Proof of Proposition 7. Let $G(\tau, s; a) \equiv s^{\frac{1+a}{2}} H(\tau, s; a)$. From (28) and (29), we have

$$
\begin{aligned}
B_t(\tau)/S_t &= \frac{G(\tau - t, s_t; 2\eta - 1)}{1 + \psi(1 - s_t^\eta)} \\
&\leq G(\tau - t, s_t; 2\eta - 1) && \text{(A.17)} \\
&\leq G(\tau - t, s_t; 1) && \text{(A.18)} \\
&\leq s_t^{-1} G(\tau - t, s_t; 1) && \text{(A.19)} \\
&\leq s_t^{-1/\varepsilon} G(\tau - t, s_t; 2/\varepsilon - 1) && \text{(A.20)} \\
&= B_{0t}(\tau)/S_{0t}.
\end{aligned}
$$

(A.17) follows from $\eta > 0$ and $s < 1$. (A.18) follows from noting that $G(\tau, s; a)$ is decreasing in a, when $a > 0$. To see this, note that for $s \in (0, 1)$,

$$
\frac{\partial G(\tau, s, a)}{\partial a} = \frac{1}{4} s^{1/2 - a/2} \log(s) \mathcal{G}(s) \leq 0
$$

which follows from

$$
\mathcal{G}(s) = s^a \text{Erf}\left(\frac{2\log(s) + a(\varepsilon\sigma_\delta)^2\tau}{2\sqrt{2}\varepsilon\sigma_\delta\sqrt{\tau}}\right) + s^a + \text{Erf}\left(\frac{-2\log(s) + a(\varepsilon\sigma_\delta)^2\tau}{2\sqrt{2}\varepsilon\sigma_\delta\sqrt{\tau}}\right) - 1
$$

with

$$
\begin{aligned}
&\mathcal{G}(0) = 0, \\
&\mathcal{G}(1) = 2\text{Erf}\left(\frac{a\varepsilon\sigma_\delta\sqrt{\tau}}{2\sqrt{2}}\right) \geq 0, \\
&\mathcal{G}'(s) = a s^{a-1}\left(1 + \text{Erf}\left(\frac{2\log(s) + a(\varepsilon\sigma_\delta)^2\tau}{2\sqrt{2}\varepsilon\sigma_\delta\sqrt{\tau}}\right)\right) \geq 0.
\end{aligned}
$$

(A.19) is implied by the fact $s \in (0, 1)$. (A.20) follows by direct differentiation. In particular, note that $s_t^{-a} G(\tau - t, s_t; 2a - 1)$ is increasing in a. ■

Proof of Proposition 8. The portfolio decomposition follows from an application of Itô's lemma to the expression in (31) and matching diffusion terms. In particular, (32) and (33) solve the system

$$
\phi_{1t}^S(\tau) v(s_t) + \phi_{1t}^{B_0}(\tau) \Sigma_t^0(\tau) = \frac{1 - (1 - \varepsilon)s_t}{1 + \gamma} P_t, \qquad \phi_{1t}^S(\tau) + \phi_{1t}^{B_0}(\tau) = \frac{1 - s_t}{1 + \gamma} P_t.
$$

The sign of $\phi^S(\tau)$ follows from the fact that $\Sigma_t^0(\tau) < 0$. The sign of $\phi^{B_0}(\tau)$ follows from

$\text{sign}\left[\phi^{B_0}\right] = \text{sign}\left[\phi_1\right]$. ■

Proof of Corollary 1. The volatility of the stock follows from an application of Itô's lemma to (25). ■

Proof of Proposition 9. From (36), the expected utility of the arbitrageur is given by

$$U_3(\psi) = U_0 + \underbrace{\rho^{-1}\log\left(\frac{\gamma}{1+\gamma}\right)}_{\text{(A)}} + \underbrace{E\left[\int_0^\infty e^{-\rho t}\log(1-s_t)dt\right]}_{\text{(B)}}.$$

We compute the derivative of (A, B) with respect to ψ. We first note that the starting point of the consumption share process is decreasing in ψ,

$$\frac{\partial s_0}{\partial \psi} = -\frac{\partial g/\partial \psi}{\partial g/\partial s_0} = -\frac{s_0^{1+\eta}\alpha}{(1+\psi)\left((1+\psi)s_0 + \alpha\eta\psi s_0^\eta\right)} < 0,$$

The term in (A) is increasing in ψ. To see this, first note that the relative weight of the arbitrageur γ is increasing in ψ,

$$\frac{\partial \gamma}{\partial \psi} = \frac{P_0(P_0(1-\alpha)-\beta)s_0^{1+\eta}}{\left((P_0(1-\alpha)-\beta)(1+\psi) + P_0(1-\alpha)\psi s_0^\eta\right)^2(s_0 + \alpha\eta\nu s_0^\eta)} > 0.$$

The numerator is positive since $P_0(1-\alpha)-\beta > 0$ is implied by (A.8). Finally, $\frac{\partial}{\partial x}\log\left(\frac{x}{1+x}\right) > 0$ for $x > 0$.

The term in (B) is increasing in ψ. To see this, we use $\frac{\partial s_0}{\partial \psi} < 0$ and the fact s is increasing in its starting point, s_0. To verify the latter, it suffices to check that the process X_t in (A.4) is increasing in its starting point. From Protter (2004) (Th. V.39), $\partial X_t/\partial X_0 = v_t$ is a strictly positive process which evolves according to $dv_t/v_t = -(1+2X_t)\varepsilon\sigma_\delta dZ_t$. ■

Proof of Proposition 10. Solving for the starting point of s_0 given (39) yields

$$s_0 = s^* \equiv \frac{\alpha(P_0 - \ell) + \beta}{P_0}.$$

We next ensure that the stock price in (39) is nonnegative,

$$\begin{aligned} S_t/P_t &= 1 - e^{qt}\frac{\ell}{P_0}\left(\frac{s_t}{s_0}\right)^{1/\varepsilon} \\ &\geq 1 - \frac{\ell}{P_0}\left(\frac{s_t}{s_0}\right)^{1/\varepsilon} \qquad \text{(A.21)} \\ &\geq 1 - \frac{\ell}{P_0}\left(\frac{1}{s_0}\right)^{1/\varepsilon} \geq 0 \qquad \text{(A.22)} \end{aligned}$$

In (A.21) use (40). (A.22) follows from $s < 1$. Initial consumption share $s^* \in (0, 1)$ and $w_1(s^*) > 0$ are guaranteed by

$$-\alpha(P_0 - \ell) < \beta < (1 - \alpha)(P_0 - \ell), \qquad \ell < P_0. \tag{A.23}$$

The inequalities in (41) follow from (A.22) and (A.23). ■

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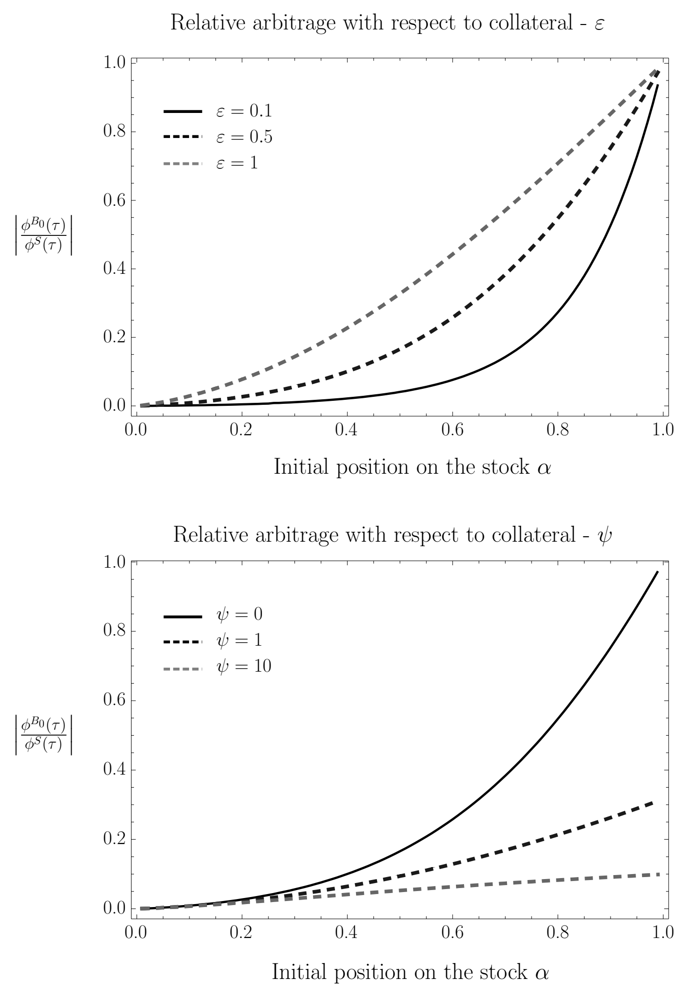

Figure 1: The figure plots the arbitrage holdings of agent 1 relative to collateral. Parameters are set to $\tau = 20$, $\delta_0 = 1$, $\sigma_\delta = 0.2$, $\rho = 0.05$, $\psi = 0$ (upper panel), $\varepsilon = 0.5$ (lower panel). The upper panel shows that the unconstrained agent increases his arbitrage position per unit of collateral as $\varepsilon \uparrow$ (the volatility of B_0 goes down with ε, the stock volatility is fixed at $\sigma_t = \sigma_\delta$). The lower panel shows that the unconstrained agent reduces his arbitrage position per unit of collateral as $\psi \uparrow$ (the stock volatility goes up with ψ)

.

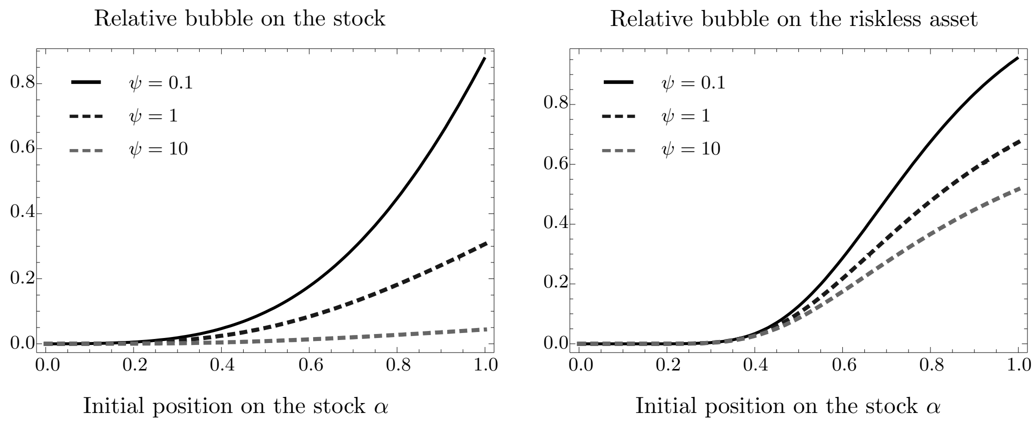

Figure 2: The left (right) panel plots the relative size of the bubble on the stock $B_t(\tau)/S_t$ (on the riskless asset $B_{0t}(\tau)/S_{0t}$) for different levels of ψ. Parameters are set to $\tau = 20$, $\delta_0 = 1$, $\sigma_\delta = 0.2$, $\rho = 0.05$, $\varepsilon = 0.5$.