Cash Flow Multipliers and Optimal Investment Decisions

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ABSTRACT: Valuation multipliers are frequently used in practice. We analyze the multiplier that yields the value of the firm when multiplied by the current cash flows of the firm, i.e. the cash flow multiplier. By postulating a simple stochastic process for the firm's cash flows in which the drift and the variance of the process depend on the investment policy, we develop a stylized model that links the cash flow multiplier to the optimal investment policy. Our model implies that the multiplier increases with investment at a decreasing rate (diminishing marginal returns on capital), i.e. there is a nonlinear relationship between the multiplier and investment. On the other hand, the multiplier is inversely related to discount rates. Using an extensive data set we examine the implications of our model. We find strong support for the variables postulated by the model.

KEYWORDS: Firm valuation, Valuation multiples, Real options

JEL-CLASSIFICATION: C61, G12, G13, M40

Valuation multiples are frequently used in practice since they offer a quick way to value a firm without estimating the whole series of future cash flows. This paper offers a parsimonious model that relates the value of a firm to its current cash flow. Postulating a simple stochastic process for the firm's cash flows (before investment) in which the drift and the variance of the process depend on the investment policy of the firm, we derive a closed-form solution of the cash flow multiplier that takes into account optimal investment and how this investment affects future cash flows.We show how the cash flow multiplier is (negatively) related to the discount rate and (positively related) to optimal investment. Furthermore, it is shown that the multiplier depends on optimal investment in a nonlinear way. Then this multiplier is decomposed into two parts: the first part reflects the firm value without investment, whereas the second part captures the option to invest optimally in the future.

Using a data set comprised of more than 16,500 firms over 38 years we examine the different predictions of our model using macro and firm-specific explanatory variables. Both types of variables can be subdivided into variables that are part of our model and variables that are used as controls. For instance, we include macro variables that affect the discount rate such as the real short-term interest rate, the slope of the term structure of interest rates, and a credit spread (spread of Baa bonds over Treasuries). Increases in all of these variables have a positive effect on the discount rate, and therefore should have a negative effect on the cash flow multiplier. Besides, we add inflation and the volatility of the S&P500 index to control for the state of the economy. As firm specific variables we include the proportion of cash flows invested that comes directly from our theoretical model and should, if investment is optimal, be positively related to the cash flow multiplier. We also run regressions where we include a squared term to check the model prediction that the multiplier increases with investment at a decreasing rate. In most of the regressions we include size, leverage, and a dividend dummy as control variables as well as firm and/or industry fixed effects. Besides, we study the effect of R&D expenses that are part of a firm's investment policy, but that are missing in about 50% of the observations. We find that all the explanatory variables related to the discount rate - the real short term interest rate, the slope of the term structure, and the spread of Baa bonds over Treasuries - have the correct sign and most of them are significantly negative. The proportion of cash flow invested is always highly significant and positive as predicted by the model. We also provide empirical evidence that the relationship is nonlinear and relate the cash flow multiplier to the average size of an industries' investment policy.

Furthermore, firms in certain industrial sectors require more investment because obsolescence in the sector is faster, because the sector is more competitive, or because the sector is more heavily regulated. In our theoretical model the drift of the cash flow process (without investments) can proxy for this phenomenon. The smaller (or more negative) is this drift without investments, the more investment will be required to keep or increase the level of future cash flows. This would imply that, even though the cash flow multiplier for a given firm is positively related to the proportion of its cash flow invested, the multiplier should be negatively related to the average investment proportion of the industry to which it belongs since it would be a more intensive investment industry. We find evidence of this in the data as well.

Our paper is related to several strands of the literature. First it contributes to an extensive literature on multiples. For instance, Baker and Ruback (1999) study how to estimate industry multiples and how to choose a measure of financial performance as a basis of substitutability. They find that EBITDA is a better single basis of substitutability than EBIT or revenue. They analyze the valuation properties of a comprehensive list of multiples and also examine related issues such as the variation in performance across industries and over time. Liu, Nissim, and Thomas (2002) analyze the valuation properties of a comprehensive list of multiples. They also examine related issues such as the variation in performance across industries and over time. This analysis is extended by Liu, Nissim, and Thomas (2007). Bhojraj and Ng (2007) examine the relative importance of industry and country membership in explaining cross-sectional variation in firm multiples. These papers, however, do not include macroeconomic variables in the analysis.

In a series of papers, Ang and Liu (2001, 2004, 2007a) address theoretical issues related to our paper. For example, Ang and Liu (2001) derive a model that relates firm value to accounting data under stochastic interest rates, heteroskedasticity and adjustments for risk aversion. Ang and Liu (2004) develop a model that consistently values cash flows with changing riskfree rates, predictable risk premiums, and conditional betas in the context of a conditional CAPM. Finally, Ang and Liu (2007a) show theoretically that a given dividend process and any of the variables – expected return, return volatility, and the price-dividend ratio – determines the other two. Although they do not model investment decisions explicitly, they derive a partial differential equation for the price-dividend ratio that is also satisfied by the cash flow multiplier in our paper given that the corresponding firm invests optimally. Therefore, our model can be thought of as an extension of their model endogenizing a firm's investment decision. Furthermore, Brennan and Xia (2006) analyze the risk characteristics and valuation of assets in an economy in which the investment opportunity set is described by the real interest rate and the maximum Sharpe ratio. They study the betas of the securities and the discount rates of risky cash flows in their model.

Berk, Green, and Naik (1999) develop a model that values the firm as the sum of the present value of its current cash flows and its growth options and is thus similar in spirit to the theoretical model in our paper. But their main interest is to study the dynamics for conditional expected returns. In a related paper, Carlson, Fisher, and Giammarino (2004) derive two theoretical models that relate endogenous firm investment to expected return. Titman, Wei, and Xie (2004) document a negative relation between abnormal capital investments and future stock returns. Anderson and Garcia-Feijoo (2006) find in an empirical study over the time period from 1976 to 1999 that growth in capital expenditures explains returns to portfolios and the cross section of future stock returns. Furthermore, Li and Zhang (2009) and Li and Zhang (2010) study the cross section of stock returns in investment-based asset pricing frameworks. Xing (2009) finds that in the cross-section portfolios of firms with low investment-to-capital ratios have significantly higher average returns than those with high investment-to-capital ratios.

Our theoretical findings concerning the value of a firm's option to invest are related to the real options literature that started with the papers by Brennan and Schwartz (1985) and McDonald and Siegel (1986). More recently, Grenadier (2002) and Aguerrevere (2009) show that in competitive markets the value of the option to invest can decrease substantially.

Finally, our paper is related to the literature that looks at the investment sensitivity of stock prices. There are a number of papers that have studied this sensitivity from a theoretical and empirical point of view, including Dow and Gorton (1997), Subrahmanyam and Titman (1999), and, more recently, Chen, Goldstein, and Jiang (2007).

The paper proceeds as follows. Section 1 develops the theoretical model. Section 2 solves for the optimal investment and the optimal cash flow multiplier and illustrates some of the implications of the model. Section 3 describes the data used in the empirical analysis and Section 4 presents the results of the panel regressions. We also include additional explanatory variables such as R&D expenses and leverage. Section 5 reports results of several robustness checks on the basic results. Section 6 provides insights into the value of the option to invest using the data available. Finally, Section 7 concludes. Some of the proofs are given in the Appendix.

1 Model

This section develops a parsimonious model where firm value naturally arises as the present value of the firm's free cash flows. Investment decisions are made endogenously affecting the expected growth rate and the volatility of the cash flow stream. The firm value is given by[1]

$$V(c,x) = \max_{\pi} \mathrm{E}\Big[\int_0^\infty e^{-\int_0^s R_u\,du}(C_s - I_s)ds\Big], \tag{1}$$

where C denotes the firm's free cash flow before investments and $I = \pi C$ denotes the dollar amount of the cash flow that is invested.[2] The variable π can be continuously adjusted by the firm and stands for the percentage of the cash flow that the firm invests. This paper is concerned with valuing a firm as present value of total cash flows and is not directly addressing the issue of the capital structure. However, if the percentage π is bigger than one, this can be interpreted as a situation where the firm uses external financing (equity or debt). The variable x denotes the initial value of a state process X that captures economic variables that impact the firm value, such as interest rates. The risk-adjusted discount rate R is assumed to be a function of this state process, i.e. with a slight abuse of notation $R = R(X)$. For illustration purposes, suppose for simplicity that X is equal to the default-free short interest rate and that the risk-adjusted discount rate is linear in the short rate, i.e.

$$R = \varphi + \psi r \tag{2}$$

with φ and ψ being constants. A simple model that would fit into this framework is the following: Suppose that the risk-adjusted discount rate is given by $R = r + \beta\lambda$, where λ is the risk premium and β is the firm's beta that is constant. If the default-free interest rate predicts the risk premium, then the premium could be linear in the interest rate, $\lambda = \overline{\lambda} + \lambda^r r$, with constants $\overline{\lambda}$ and λ^r such that $\varphi = \beta\overline{\lambda}$ and $\psi = 1 + \beta\lambda^r$ in our parametrization (2). We assume the cash flow to follow the dynamics[3]

$$dC = C[\mu(\pi, X)dt + \sigma(\pi, X)dW], \quad C(0) = c, \tag{3}$$

[1] An alternative representation using the pricing kernel is provided in a technical appendix which is available from the authors upon request.

[2] At this point, this assumption is without loss of generality. Formally, π is a an adapted process. Therefore, one can always choose $\pi = \phi/C$ where ϕ is an adapted process.

[3] This builds on the ideas of Merton (1974), Duffie and Lando (2001), and Goldstein, Ju, and Leland (2001), among others, who use lognormal models in which the firm cannot control for investment. Note that this representation does not allow for negative cash flows. In addition, this is true for all the literature dealing with multipliers.

where the expected growth rate and volatility, μ and σ, are functions of the state process and the percentage of the firm's cash flow reinvested. The process W is a Brownian motion. This specification implies the following result:

Proposition 1 (Linearity of Firm Value)**.** *Firm value is linear in the cash flow, i.e.*

$$V(c,x) = f(x)c, \tag{4}$$

where $f(x) = V(1,x)$*.*

Notice that V/c is the ratio of the current firm value over the current cash flow (for short: cash flow multiplier), which will be central in our further analysis. One can think of the cash flow multiplier as the multiple by which the current cash flow is multiplied to obtain the current firm value. In the literature on the dividend-discount model and its generalizations, usually this multiplier is assumed to be beyond the control of the firm and thus to be exogenously given. In contrast, we explicitly model the firm's opportunity to change its risk-return tradeoff by allowing the firm to control the expected growth rate and the volatility of the cash flow stream by its investment policy.[4] To illustrate our model and unless otherwise stated, we use the following specification of these parameters[5]

$$\mu(\pi,x) = \mu_0(x) + \mu_1\sqrt{\pi} + \mu_2\pi, \qquad \sigma(\pi) = \sigma_0 + \sigma_1\sqrt{\pi} + \sigma_2\pi, \tag{5}$$

where all coefficients except for μ_0 are constants and μ_0 is a linear function of the state process, $\mu_0(x) = \overline{\mu}_0 + \widehat{\mu}_0 x$. The function μ_0 characterizes the expected growth rate if the firm does not invest at all ($\pi = 0$). If the firm does not invest at all, then projects in place can become obsolete without being replaced. Therefore, we expect μ_0 on average to be negative. The parameters μ_1 and μ_2 capture the firm's ability to control its future growth rate. To avoid explosion of the model, μ_2 is assumed to be negative.[6] As we show later on, representation (5) gives a decomposition of the cash flow multiplier comparable to the one for the firm value derived by Berk, Green, and Naik (1999).

Furthermore, we allow the investment decisions to have an impact on the riskiness of the firms cash flow stream and thus the volatility σ can depend on π as well. For instance, if a firm invests in a

[4] In general, the dependence of μ and σ might be involved. For instance, nonlinear specifications can reflect frictions.

[5] Any concave function for the drift would provide the same qualitative results. This parametrization is chosen simply for computational convenience.

[6] Loosely speaking, this ensures that a transversality condition is satisfied.

new product, then both the firm's expected growth rate and the volatility of its cash flow stream might increase. Figure 1 illustrates two stylized specifications of the drift when the investment proportion is varied between zero and one. The drift starts below zero and then increases until it reaches its peak. For the lower curve the peak is reached around $\pi = 0.7$, whereas for the upper curve the peak is reached for some π that is greater than one.

[INSERT FIGURE 1 ABOUT HERE]

2 Solving for the Optimal Cash Flow Multiplier

The firm's decision problem (1) is a dynamic optimization problem that can be solved using stochastic control methods. This is the first goal of this section. We assume that the state of the economy is characterized by the short interest rate that has Vasicek dynamics[7]

$$dr = (\theta - \kappa r)dt + \eta dW_r, \tag{6}$$

where W_r is a Brownian motion that is correlated with the Brownian motion W that drives cash flows, i.e. $d < W, W_r >= \rho dt$ with constant correlation ρ. In this specification, the short interest rate captures the randomness of the discount rate. For simplicity, we assume (2) and (5).[8] In the Appendix, it is shown that the cash flow multiplier satisfies the following Bellman equation

$$\begin{aligned} 0 \;=\; & \max_{\pi}\{(\overline{\mu}_0 + \widehat{\mu}_0 r + \mu_1\sqrt{\pi} + \mu_2\pi)f + 1 - \pi - (\varphi + \psi r)f \\ & +(\theta - \kappa r)f_r + 0.5\eta^2 f_{rr} + \rho\eta(\sigma_0 + \sigma_1\sqrt{\pi} + \sigma_2\pi)f_r\}. \end{aligned} \tag{7}$$

Under the assumption that the Bellman equation is concave in π, which follows if $\mu_1 > 0$, the optimal investment strategy of the firm is given by

$$\pi^* = \left(\frac{\mu_1 f + \rho\eta\sigma_1 f_r}{2(1 - \mu_2 f - \rho\eta\sigma_2 f_r)}\right)^2. \tag{8}$$

Substituting the optimal investment level back into the Bellman equation (7) leads to a differential equation for the cash flow multiplier

$$0 = (\widehat{\varphi} + \widehat{\psi} r)f + 1 + (\theta + \rho\eta\sigma_0 - \kappa r)f_r + 0.5\eta^2 f_{rr} + \frac{(\mu_1 f + \rho\eta\sigma_1 f_r)^2}{4(1 - \mu_2 f - \rho\sigma_2\eta f_r)}, \tag{9}$$

[7] A generalization to two state variables can be found in a technical appendix that is available from the authors upon request.

[8] Otherwise, the corresponding Bellman equation must be solved numerically.

where $\widehat{\varphi} = \overline{\mu}_0 - \varphi$ and $\widehat{\psi} = \widehat{\mu}_0 - \psi$ are constants. Notice that the last ratio in (9) disappears if the firm does not invest. In this case, the cash flow multiplier has the explicit solution

$$f_0(r) \equiv \int_0^\infty \widehat{\mathrm{E}}\left[e^{\int_0^s \widehat{\varphi} + \widehat{\psi} r_u \, du}\right] ds = \int_0^\infty e^{A(s) - B(s) r} \, ds, \tag{10}$$

with A and B being deterministic functions of time. The expected value $\widehat{\mathrm{E}}[\cdot]$ is taken under the measure under which the short rate has the dynamics

$$dr = (\theta + \rho\eta\sigma_0 - \kappa r) dt + \eta d\widehat{W} \tag{11}$$

with $\widehat{W}$ being a Brownian motion under this measure. If the firm is investing optimally, then the last term in (9) can be thought of as an additional cash flow that the firm is able to generate by doing so. From a real option perspective, this fraction can be interpreted as the firm's option to invest optimally at a particular time t in the future. Brealey, Myers, and Allen (2010) call this the net present value of growth opportunities. The present value of this continuous series of options is given by

$$\mathcal{O}(r; f) = \int_0^\infty \widehat{\mathrm{E}}\left[e^{\int_0^s \widehat{\varphi} + \widehat{\psi} r_u \, du} \frac{(\mu_1 f(r_s) + \rho\eta\sigma_1 f_r(r_s))^2}{4(1 - \mu_2 f(r_s) - \rho\sigma_2\eta f_r(r_s))}\right] ds \tag{12}$$

such that the optimal cash flow multiplier becomes the sum of (10) and (12), i.e.

$$f(r) = f_0(r) + \mathcal{O}(r; f). \tag{13}$$

The second argument in the definition of $\mathcal{O}$ is added to emphasize that $\mathcal{O}$ depends on f. The firm has a series of options to invest and the net present value of these options is positive, $\mathcal{O} \geq 0$. The option value $\mathcal{O}$ is however not explicit since it depends on the optimal cash flow multiplier f which is unknown and a part of the solution.[9] Nevertheless, at least the first part of the representation (13) is explicitly known and equal to the solution without investing. Berk, Green, and Naik (1999) derive a similar decomposition for the firm value.[10]

To gain further insights, let us assume for the moment that the interest rate r is constant. In this case, it makes sense to simplify notations by setting $\widehat{\mu}_0 = 0$, $\varphi = \lambda = const$, and $\psi = 1$. This implies that $\mu_0 = \overline{\mu}_0 = const$ and $\widehat{\varphi} + \widehat{\psi} r = \mu_0 - r - \lambda = const$. The risk-adjusted interest rate is the sum of the short rate and a risk premium, i.e. $R = r + \lambda$. Furthermore, the optimal cash flow multiplier f is a constant and (13) simplifies into

$$f = \int_0^\infty e^{(\mu_0 - r - \lambda)s} \, ds + \underbrace{\int_0^\infty e^{(\mu_0 - r - \lambda)s} \frac{(\mu_1 f)^2}{4(1 - \mu_2 f)} \, ds}_{= \mathcal{O}(f)}, \tag{14}$$

[9] From a formal point of view, (13) is a fixed point problem for f.

[10] See equation (32) in their paper.

where the transversality condition $\mu_0 - r - \lambda < 0$ is assumed to hold. Notice that, by (8), equation (14) can be rewritten as follows $f = f_0 + f_0(1 - \mu_2 f)\pi^*$, where $f_0 = \int_0^\infty e^{(\mu_0 - r - \lambda)s}\, ds$ is the cash flow multiplier without investment. In this special case, one can see some parallels to Berk, Green, and Naik (1999). The risk premium λ captures the systematic risk and $\int_0^\infty e^{(\mu_0 - r)s}\, ds$ can be interpreted as perpetual, riskless consol bond where the payments on this bond depreciate at the a constant rate $\mu_0 < 0$. One difference however is that in their model the firm has a series of given European options to invest, whereas in our paper the firms endogenously controls the fraction of cash-flows to be invested, which makes the representation of the cash flow multiplier implicit.

Solving for the optimal multiplier f and taking logarithms yields

$$\ln f = \ln f_0 + \ln(1 + \pi^*) - \ln(1 + f_0 \mu_2 \pi^*), \tag{15}$$

which establishes a nonlinear relationship between the cash flow multiplier and the investment strategy π^* that we will use later on. The following proposition shows that under a mild condition a unique cash flow multiplier exists.

Proposition 2 (Cash Flow Multiplier under Constant State Process)**.** *If $\mu_1^2/4 - \mu_2(\mu_0 - r - \lambda) < 0$, then the optimal cash flow multiplier is uniquely given as the positive root of the quadratic equation*

$$0 = \left[\mu_1^2/4 - \mu_2(\mu_0 - r - \lambda)\right] f^2 + (\mu_0 - r - \lambda - \mu_2) f + 1. \tag{16}$$

Notice that in the special case when the firm has no control over the expected growth rate of its cash flow stream ($\mu_1 = \mu_2 = 0$), relation (16) becomes a linear equation with solution $f = 1/(r + \lambda - \mu_0)$. This is a version of the Gordon growth model. Furthermore, due to the transversality condition, a necessary requirement for the condition of Proposition 2 to hold is $\mu_2 < 0$.

To study the implications of Proposition 2, let us consider a numerical example. Similar to the previous section, we choose $\mu_0 = -0.03$, $\mu_1 = 0.1$ and $\mu_2 = -0.03$. Besides, we assume that $r = 0.04$ and $\lambda = 0.03$ such that the risk-adjusted interest rate is $R = 0.07$. The positive root of (16) which is the cash flow multiplier equals 13.06. If the firm suboptimally decides not to invest, then the option value in (14) is zero and the cash flow multiplier is 10. Therefore, the option value equals $\mathcal{O} = 3.06$. Put differently, the opportunity to invest increases the cash flow multiplier by 30 percent. Let us consider a second example where all parameters are the same as in the first example except for μ_0 which is assumed to be -0.05. As briefly discussed above, one reason for this lower value might be that the industry requires more investments to maintain its cash flow level. The cash flow multiplier resulting from optimal investing is now 9.91, whereas the cash flow multiplier without investing is 8.33. Therefore, the option value becomes $\mathcal{O} = 1.58$ or 18% of the optimal cash flow multiplier. This suggests that in an investment intensive industry the real option to invest loses value both in absolute as well as in relative terms. In fact, we are able to show that this is in general true.

Theorem 3 (Value of the Option to Invest). *If, in addition to the assumption of Proposition 2, condition $\mu_0 - r - \lambda - \mu_2 < 0$ holds, then the optimal cash flow multiplier f, the option value $\mathcal{O}$, and the ratio $\mathcal{O}/f$ are increasing in μ_0.*

Remark. The requirement $\mu_0 - r - \lambda - \mu_2 < 0$ is a bit stronger than the transversality condition since $\mu_2 < 0$. Nevertheless, it is satisfied for reasonable parametrizations of the model.

Put differently, the previous theorem says that the option's absolute and relative values decrease if μ_0 becomes more negative. This result puts some of the classical results on real options into perspective and it is related to Grenadier (2002) and Aguerrevere (2009): If the firm is forced to invest for instance because competitors do the same, then the option to invest loses (part of) its value. Hence, the cash flow multiplier decreases.

Finally, we establish the relation of the cash flow multiplier and a stochastic short rate. Then, the presence of the fraction in (9) turns the differential equation into a highly nonlinear equation, which makes solving the equation more challenging.[11] Nevertheless, there is an explicit power series representation.

Theorem 4 (Optimal Cash Flow Multiplier under Stochastic Interest Rates). *The cash flow multiplier has the following series representation*

$$f(r) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} a_i^{(n)} \left(r - \frac{\theta}{\kappa} \right)^i \eta^n \tag{17}$$

where the coefficients $a_i^{(n)}$ satisfy an explicit recursion provided in Appendix B.

To illustrate the implications of the previous theorem we consider a particular firm, Coca Cola. We calculate the cash flow multiplier over the time period from 1971 to 2008, where the relevant information about Coca Cola comes from Compustat.[12] We choose the parameters of the riskfree

[11] At least this is so if there are two state variables since in this case the equation is a nonlinear partial differential equation, a case treated in the Appendix.

[12] We postpone a detailed description of the data and the definition of the variables to Section 3.

short rate process (6) to be $\kappa = 0.08$, $\eta = 0.015$, and $\theta = 0.004$. This implies that the mean reversion level is $\theta/\kappa = 0.05$, which is about the sample average of the one-month Fama-French riskfree rate as reported by CRSP.[13] For Coca Cola we have 38 observations of the cash flow multiplier observed on December 31 of the particular year. Figure 2 plots the logarithms of these observations against the realizations of the Fama-French one-month riskfree rate. Obviously, both variables are negatively related. This figure also depicts the least-square fit of our model which is almost linear. Various numerical experiments have shown that this is a general feature of our model.

[INSERT FIGURE 2 ABOUT HERE]

3 Data

Our sample period covers 38 years ranging from 1971 to 2008. The data comes from several sources. The first is the combined annual, research, and full coverage 2008 Standard and Poor's Compustat industrial files. The sample is selected by first deleting any firm-year observations with missing data. The only exception is deferred taxes (Compustat txdb) that we set to zero if it is missing. The reason is that deferred taxes are typically an insignificant part of firm value compared to the book and market value of the assets (at, csho, prcc_f, and ceq) and we would have lost around 10% of the observations if we had deleted them. To check the robustness of this assumption, we run our benchmark regression (1) that is reported in Table 4 excluding all observations where deferred taxes are missing. As expected, the results are virtually unchanged. In our analysis we use the following definition of free cash flows (before investment): We take the difference between EBITDA and taxes (oibdp minus txt), subtract the change in working capital (annual change in wcap) and add asset sales (sppe). The cash flow multiplier is then defined as the ratio between firm value and free cash flows where firm value is given by sum of book value plus the difference between market value and book value of equity minus deferred taxes.[14] The concept of valuation multiples cannot be applied if the corresponding value driver is negative.[15] Therefore, we disregard negative realizations of the

[13]The reason for not formally estimating θ and κ over the period from 1971 to 2008 is that there are regime shifts during this period like the spike in 1979 that cannot be well calibrated by a one-factor model.

[14]Firm value is thus given by at + prcc_f $\times$ csho - ceq - txdb, where we set deferred taxes equal to zero if they are missing.

[15]Similar problems can occur with negative earnings that make payout ratios ill-defined. See, e.g., Li and Zhang (2010).

cash flow multipliers so that our sample consists of 108,443 observations stemming from 16,567 firms.

We use several macroeconomic explanatory variables: The one-month Fama-French riskfree rate was obtained from CRSP. Since data on expected inflation is not easily available over the time period from 1971 to 2008 (e.g. TIPS are traded since 1997 only), we use the realized inflation over the previous 12 months as reported by the Bureau of Labor Statistics.[16] However, as inflation is not only measuring price changes, but is also related to the state of the economy (e.g. high inflation might create uncertainty), in the regressions we control for this second feature by regressing on inflation as well. The real riskfree rate is then approximated by the difference between the Fama-French riskfree rate and this inflation rate. The Treasury yields and the corporate bond yields are from Global Financial Data. The slope of the Treasury yield curve is defined as the difference between the 14 year Treasury yield and the riskfree rate. The 14 year Treasury yield is obtained by linearly interpolating between the 10 year and 20 year yields. We use this maturity since it can be matched against the industrial Baa corporate bond yields reported by Moody's to calculate the 14 year yield spread between corporate bonds and Treasury bonds.[17] Besides, we calculate the historical volatility of the stock market from the value weighted S&P 500 index as reported in CRSP. We use the version without dividends, but the volatilities obtained from the version with dividends are almost identical. We include the last 250 trading days to compute the volatility. Finally, we include the relative value of the S&P 500 index. More precisely, we use the log of the S&P 500 index and subtract its trend that was observed from 1926 to the end of 1970. Table 1 presents the summary statistics of the macro variables measured at the last trading day of a particular month. All variables are annualized and quoted in percentages. For instance, the average riskfree rate over the period from 1971 to 2008 was 5.565%. The maximum of 16.15% was reached in May of 1980 and the minimum of 0.03% in November and December of 2008. The yield spread reached its maximum of 5.788% in November of 2008 and its minimum of 0.754% in October of 1978. Finally, the average annualized historical volatility is about 14.8%. Since a year has about 250 trading days, we multiply the daily volatility by the square root of 250 to obtain the annual volatility.

[INSERT TABLE 1 ABOUT HERE]

In the empirical analysis to follow, we regress the logarithm of the cash flow multiplier on several

[16] More precisely, we use the 12 month consumer price index - all urban consumers.

[17] The average maturity of the bonds in the index provided by Moody's is about 14 year.

variables. The first three are closely related to the term structure and include the real riskfree rate, the slope of the Treasury term structure, and the spread of Baa rated bonds over Treasury bonds. We decided to include the real riskfree rate (instead of the nominal one) since one could argue that the model (1) is set up in real terms. Holding the other variables fixed, an increase in either of these variables increases the discount rate at which free cash flows are discounted. Since in our model the discount rate is negatively related with the multiplier, we expect to observe negative relations between the multiplier and these variables. We also include the historical volatility of the S&P 500 as an explanatory variable measuring aggregate equity market risk. We expect to observe a negative relationship between volatility and the multiplier. The same is true for inflation that is also used as a control variable. As last macro variable that controls for the relative value of the stock market we include the detrended log-value of the S&P 500 index. We expect the cash flow multiplier to be positively related to this measure.

The first-order condition (8) of our model suggests that the multiplier increases with the proportion of the cash flows invested. To test this prediction empirically, we add a proxy for this variable to the set of our explanatory variables. We measure the investment proportion by the ratio of the annual capital expenditures (Compustat capx) over the free cash flows. This variable is winsorized at the 1% level. We will later study the sensitivity of the results to this procedure. To control for size effects, we include the logarithm of the real market capitalization as an explanatory variable. The market capitalization of a firm is defined as the product of the number of shares outstanding and the price per share (Compustat csho and prcc_f). Real market capitalization is then calculated by dividing market capitalization by the consumer price index (CPI). We also control for whether a firm pays a dividend by adding a dividend dummy. Finally, we take a firm's leverage into account. We use the ratio of long-term debt (Compustat dd1 plus ddltt) to firm value, which is intended to measure the likelihood of distress. Roll, Schwartz, and Subrahmanyam (2009), among others, argue that firms which have more leverage are riskier and therefore require a higher discount rate. This would lead to a negative sign of the regression coefficient. Alternatively, Jensen and Meckling (1976) argue that debt might make managers more careful about investments, which would result in a positive sign of the regression coefficient. Table 2 presents the summary statistics of the firm specific variables and Table 3 summarizes the correlations between all firm specific and all macro variables. Note that the highest correlation in the table is between volatility and the Baa spread (0.71); both of them represent some measure of global risk. The second highest correlation in the table is between the log of the cash flow multiplier and the proportion of the cash flow invested, π, which are the two key variables of interest in our analysis.

[INSERT TABLES 2, 3 ABOUT HERE]

4 Panel Regression Results

In this section we examine the determinants of the cash flow multiplier by running several panel regressions that use all the information contained in the time-series. The residuals of the cross-sectional regressions are likely to be serially correlated. Furthermore, as we will demonstrate later on, there might be cross-sectional dependance as well. To overcome these potential problems, we correct our t-statistics using the approach outlined in Driscoll and Kraay (1998). They assume an error structure that is heteroscedastic, autocorrelated up to some lag, and possibly correlated between the units.[18] The resulting standard errors are heteroscedasticity consistent as well as robust to very general forms of cross-sectional and temporal dependence. In our robustness checks, we will discuss this point in more detail.

[INSERT TABLE 4 ABOUT HERE]

Our benchmark result (1) is a fixed-effects regression presented in Table 4. As postulated in the previous section, the real riskfree rate, the Baa spread, inflation, and volatility have significantly negative impacts on the cash flow multiplier. The slope of the term structure and the relative value of the S&P 500 are insignificant. The firm specific variables are all very significant and have the expected signs. Besides, as predicted by the model, the fraction of the cash flows invested, π, is very significant and positively related with the multiplier. The same is true for the size measure, which serves as a control variable. On the other hand, in line with the literature, the remaining two control variables, leverage and the dividend dummy, have negative loadings.[19]

Column (2) in Table 4 reports results when we run a regressions with dummies for the 48 Fama-French industries (instead of firm fixed effects).[20] The significance levels of the significant coefficients remain almost the same as before. However, the real riskfree rate is borderline significant at the 5% level now and the spread is significant at the 10% level only. In order to test for the

[18] In our regressions, the maximum lag is three years.

[19] See, e.g., Roll, Schwartz, and Subrahmanyam (2009).

[20] Excluding regulated, financial, or public service firms (Fama-French industries 31 and 48 as well as Fama-French industries 44-47) does not change our results.

presence of firm-specific fixed effects, we performed a robust version of the Hausman test.[21] The null hypothesis of no fixed-effects is rejected at all levels suggesting that there are fixed effects in the data. This is the reason why we have chosen regression (1) to be our benchmark regression. Unless otherwise stated, in the sequel we thus report the results of fixed-effects regressions.

Furthermore, we consider regressions where we exclude some of the explanatory variables. The results are reported in Table 5. Comparing regressions (3) with regression (4) shows that volatility and the spread variable are substitutes: If we take out one of them, then the remaining variable becomes highly significant. Furthermore, the explanatory power of the macro variables is small as regression (5) demonstrates. Only volatility is significant. On the other hand, the firm specific variables can explain most of the variation in the cash flow multiplier. The loadings and their significance levels remain almost unchanged if we take out the macro variables, which can be seen in regression (6). In particular, the fraction of cash flows invested, π, explains a large part of the variation in the multiplier (see (7)).

[INSERT TABLE 5 ABOUT HERE]

Finally, we study the functional relationship between the cash flow multiplier and the investment fraction π. Equation (15) can be approximated by applying the Taylor approximation $\ln(1+x) \approx x - 0.5x^2$

$$\ln f \approx \ln f_0 + \beta_1 \pi + \beta_2 \pi^2, \tag{18}$$

where β_1 is positive and β_2 is negative. This means that the multiplier increases with π at a decreasing rate (decreasing marginal productivity of capital), i.e. there is a nonlinear relationship between the multiplier and π. Regressions (8) and (9) of Table (6) confirm this conjecture. Note that the nonlinear effect is small, but it is still very significant.

[INSERT TABLE 6 ABOUT HERE]

5 Robustness Checks

In this section, we report the results of several checks on the basic results. The tests consider standard errors and additional explanatory variables. We first compare the standard errors of regression (2) with the standard errors that obtain if we form clusters by firm and year (regression

[21] See, e.g., Wooldridge (2002), p. 288ff.

(10)) or by firm only (regression (11)).[22] The results are reported in Table 7. Notice that the point estimates for the first two regressions are exactly the same. Besides, the p-values of our benchmark regression (1) are smaller than in regressions (10) and (11). This suggests that there is spatial correlation in the data.[23] Besides, clustering by firm only clearly leads to overly optimistic standard errors. The same would be true if we run a fixed-effects regression clustering by firm only.

[INSERT TABLE 7 ABOUT HERE]

Next, we study whether the level where we winsorize the investment fraction matters. To analyze this question, we alternatively run regression where we winsorize π either at the 5% level or set π to one if it is larger than one. The results are presented in Table 8. As can be seen the significance levels of the variables are hardly affected. In particular, the fraction of the cash flows invested remains highly significant. The loadings on π however increase, since the upper bounds on π are smaller in regressions (12) and (13). This is also reflected in the mean of π that is reported in the table. Notice that winsorizing does not affect the median as can be seen as well.

[INSERT TABLE 8 ABOUT HERE]

Our proxy for investments are capital expenditures (Compustat capx) that do not include R&D expenses (Compustat xrd). The main reason for using this proxy is that otherwise we would have lost more than 50% of our observations since xrd is often missing in Compustat. To check whether including R&D expenses changes our results, we run regressions where we set R&D expenses to zero if they are missing. We then add R&D expenses to the capital expenditures whenever they are positive. In this case, the number of observations remains the same as in the benchmark regression. The results are given in column (15). The real riskfree rate and inflation are now significant at the 10% level only, whereas the remaining significance levels are as before. Alternatively, one can define a second investment ratio for R&D expenses. Firstly, we set this ratio to zero whenever R&D expenses are missing. Column (16) shows that the results are almost identical to our benchmark regression (1). Additionally, the investment fraction for R&D expenses is highly significant and positive. Finally, we have run regression (17) where we disregard observations if R&D expenses are missing. This regression is based on 53,887 observations coming from 9,291 firms. Again both

[22] See, e.g., Pedersen (2009) and the references therein.

[23] To test for spatial correlation, we performed Pesaran's test of cross sectional independence on a subsample of firms with at least 30 observations. This test rejects independence at all significance levels, which suggest that Driscoll-Kraay standard errors are more appropriate.

investment ratios are highly significant and have positive coefficients. Besides, the levels of the other firm specific variables are not affected as well. For the macro variables the levels decrease so that the real riskfree rate is only borderline significant at the 10% level and inflation is not significant any more.

[INSERT TABLE 9 ABOUT HERE]

6 Value of the Option to Invest

We have shown that the cash flow multiplier consists of two parts (see, e.g., (13)): Whereas the first part is exogenous, the second part is endogenous and captures the firm's real option to invest, the so-called present value of growth opportunities. Besides, Theorem 3 proves that the option value is increasing with μ_0. This parameter equals the expected cash flow growth if the firm does not invest at all. We expect μ_0 to be on average smaller when the firm operates in an industry that is more investment intensive. The investment intensity of an industry is measured by the average fraction of cash flows that is reinvested, i.e. by the average π of a particular industry. To test this hypothesis, we run regressions where this average is included as an additional explanatory variable. We have already seen that the cash flow multiplier increases with π. Following our line of argument, the opposite should be true for the mean of the industry. There are two ways of calculating an industry mean. Firstly, one can calculate the mean over the whole sample period leading to a constant. Secondly, one can compute the mean for every year of the sample period, which provides us with 48 time series of means for the 48 Fama-French industries. In the first case, it clearly makes no sense to include firm dummies or fixed effects since otherwise the coefficients of the average π cannot be identified. But also in the second case dummies would absorb a lot of the variability that we expect to be captured by the industry means of π. For this reason, we run pooled regressions without dummies and report the results in Table 10. Regression (18) includes the same explanatory variables as the benchmark regression (1). The variable Av_pi denotes the average π of the corresponding industry over the whole sample period of 38 years. In contrast, Av_pi_annual denotes the average π of the corresponding industry calculated every year leading to 48 time series. It can be seen that in all regressions the coefficients on the average π are significantly negative, which supports our line of argument above. The significance levels of the other variables are almost unchanged. The real riskfree rate and inflation are however only significant at the 10% level.

[INSERT TABLE 10 ABOUT HERE]

7 Conclusion

We develop a simple discounted cash flow valuation model with optimal investment. The model predicts a positive relation between the cash flow multiplier and a firm's investment policy and a negative relation between the multiplier and discount rates. These predictions are confirmed in our empirical analysis where we include additional macro and firm specific control variables. Our panel regression results indicate that the explanatory variables have the correct sign and for the most part are highly significant. Our model also implies that the relation between the multiplier and investment is nonlinear. We provide empirical evidence that this is the case. Besides, we decompose the multiplier into two parts: the first part reflects the firm value without investment, whereas the second part captures the option to invest optimally in the future. We provide empirical evidence that the cash flow multiplier is strongly negatively related to the average investment policy of the particular industry.

Since the cash flow multiplier depends on observable and relatively easily obtainable variables, the approach taken in this paper could be easily used in practice. Even though it is based on a discounted cash flow model it does not require the estimation of expected future cash flow and an appropriate risk-adjusted discount rate.

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A Proofs

Proof of Proposition 1. We set $Y_t = C_t/c$ such that $Y_0 = 1$. Then, problem (1) can be rewritten as

$$\begin{aligned} V(c,x) &= \max_\pi \mathrm{E}\Big[\int_0^\infty e^{-\int_0^s R_u\,du}(cY_s - \pi_s cY_s)ds\Big] \qquad (19)\\ &= c\max_\pi \mathrm{E}\Big[\int_0^\infty e^{-\int_0^s R_u\,du}(Y_s - \pi_s Y_s)ds\Big] = cV(1,x). \end{aligned}$$

This implies $V_c(c,x) = V(1) = const$ and $V_{cc}(c,x) = 0$, which shows that V is linear in c. □

Proof of the Hamilton-Jacobi-Bellman equation (7). The firm value satisfies the Hamilton-Jacobi-Bellman equation

$$0 = \max_\pi \{(\mu(\pi,r)cV_c + c - \pi c - R(r)V + (\theta - \kappa r)cV_r + 0.5\eta^2 cV_{rr} + \rho\eta\sigma(\pi)cV_{cr}\}.$$

Applying the separation (4) yields (7). □

Proof of Proposition 2. The equation (16) follows from (14). By Vieta's formulas, the two solutions, f_1 and f_2, satisfy

$$f_1 f_2 = \frac{1}{\mu_1^2/4 - \mu_2(\mu_0 - r - \lambda)}.$$

implying that there exists a unique positive cash flow multiplier if $\mu_1^2/4 - \mu_2(\mu_0 - r - \lambda) < 0$, i.e. if the parabola defined in (16) has a maximum. □

Proof of Theorem 3. We set $K = \mu_1^2/4 - \mu_2(\mu_0 - r - \lambda)$. By assumption of Proposition 2, K is negative. Due to the transversality condition, this implies that $\mu_2 < 0$. We interpret (16) as the implicit definition of f as a function of μ_0. For this reason, we interpret the right-hand side of (16) as a function F of f and μ_0. Then

$$f' = \frac{df}{d\mu_0} = -\frac{\partial F/\partial \mu_0}{\partial F/\partial f} = \frac{-\mu_2 f^2 + f}{2Kf + \mu_0 - r - \lambda - \mu_2} > 0,$$

since, by assumption, $\mu_0 - r - \lambda - \mu_2 < 0$. Next recall that

$$\mathcal{O} = \frac{-1}{\mu_0 - r - \lambda}\frac{\mu_1^2 f^2}{4(1-\mu_2 f)}.$$

Therefore,

$$\mathcal{O}' = \frac{d\mathcal{O}}{d\mu_0} = \frac{1}{(\mu_0 - r - \lambda)^2}\frac{\mu_1^2 f^2}{4(1-\mu_2 f)} + \frac{-1}{\mu_0 - r - \lambda}\frac{\mu_1^2}{4}\frac{ff'(2-\mu_2 f)}{(1-\mu_2 f)^2} > 0.$$

Finally, $\frac{\mathcal{O}}{f} = \frac{\mathcal{O}}{\frac{-1}{\mu_0 - r - \lambda} + \mathcal{O}}$. Consequently,

$$\frac{d}{d\mu_0}\frac{\mathcal{O}}{f} = \frac{-1}{\mu_0 - r - \lambda}\frac{\mathcal{O}' - \frac{1}{\mu_0 - r - \lambda}\mathcal{O}}{f^2} > 0,$$

since $\mathcal{O}' - \frac{1}{\mu_0 - r - \lambda}\mathcal{O} = \frac{-1}{\mu_0 - r - \lambda}\frac{\mu_1^2}{4}\frac{ff'(2-\mu_2 f)}{(1-\mu_2 f)^2} > 0$. This completes the proof of Theorem 3. □

B Series Expansion of Theorem 4

We firstly provide a representation of the coefficients in the series expansion (17) and then prove Theorem 4). We define $\widehat{\theta} = \theta/\kappa$, $\widetilde{\varphi} = \widehat{\varphi} + \widehat{\psi}\widehat{\theta}$, and $H_{i,j}^{k,\nu} = \widetilde{a}_i^{(k)} b_j^{(\nu)} + 0.25 c_i^{(k)} c_j^{(\nu)}$, where

$$\begin{aligned}
\widetilde{a}_i^{(\nu)} &= \mathbf{1}_{\{\nu=i=0\}} - \mu_2 a_i^{(\nu)} - \rho\sigma_2 (i+1) a_{i+1}^{(\nu-1)}, \\
b_i^{(\nu)} &= \mathbf{1}_{\{\nu=i=0\}} + \widetilde{\varphi} a_i^{(\nu)} + \widehat{\psi} a_{i-1}^{(\nu)} - \kappa i a_i^{(\nu)} + \rho\sigma_0 (i+1) a_{i+1}^{(\nu-1)} + 0.5(i+2)(i+1) a_{i+2}^{(\nu-2)} \\
c_i^{(\nu)} &= \mu_1 a_i^{(\nu)} + \rho\sigma_1 (i+1) a_{i+1}^{(\nu-1)}.
\end{aligned} \tag{20}$$

Then the coefficients are given by the following explicit recursion

$$\begin{aligned}
a_0^{(n)} &= -\frac{\sum_{k=1}^{n-1} H_{0,0}^{k,n-k} + R_0^{(n)}}{D_0}, \\
a_m^{(n)} &= -\frac{\sum_{(i,k)\in\mathcal{I}} H_{i,m-i}^{k,n-k} + R_m^{(n)}}{D_m},
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
R_m^{(n)} &= (1 - \mu_2 a_0^{(0)}) \left[\mathbf{1}_{\{m=n=0\}} + \widehat{\psi} a_{m-1}^{(n)} + \rho\sigma_0 (m+1) a_{m+1}^{(n-1)} + 0.5(m+2)(m+1) a_{m+2}^{(n-2)}\right] \\
&\quad + (1 + \widetilde{\varphi} a_0^{(0)}) \left[\mathbf{1}_{\{m=n=0\}} - \rho\sigma_2 (m+1) a_{m+1}^{(n-1)}\right] + 0.5\mu_1\rho\sigma_1 (m+1) a_0^{(0)} a_{m+1}^{(n-1)}, \\
D_m &= (1 - \mu_2 a_0^{(0)})(\widetilde{\varphi} - m\kappa) - \mu_2 (1 + \widetilde{\varphi} a_0^{(0)}) + 0.5\mu_1^2 a_0^{(0)}
\end{aligned} \tag{22}$$

and $\mathcal{I} = \{0, 1, \ldots, m-1, m\} \times \{0, 1, \ldots, n-1, n\} \setminus \{(0,0), (m,n)\}$ is an index set.[24]

We emphasize that this recursion is explicit and all equations (21) do not involve $a_m^{(n)}$ on the right-hand side. The only exception is the equation for $a_0^{(0)}$ where $a_0^{(0)}$ appears on the left- and right-hand side. This leads to the following quadratic equation.

$$0 = (0.25\mu_1^2 - \mu_2\widetilde{\varphi})(a_0^{(0)})^2 + (\widetilde{\varphi} - \mu_2) a_0^{(0)} + 1.$$

[24]Therefore, the difference $\sum_{k=0}^{n}\sum_{i=0}^{m}\cdots$ has two more elements than $\sum_{(i,k)\in\mathcal{I}}\cdots$, namely the elements with indices $(i,k) = (0,0)$ and $(i,k) = (m,n)$.

For reasonable parametrizations, numerical experiments suggest that this equation has one positive and one negative root.

Proof of Theorem 4. We firstly multiply equation (9) by $4(1-\mu_2 f-\rho\sigma_2\eta f_r)$. Then we substitute the representation (17) into the resulting equation. This leaves us with several products of power series. Expanding these products and rearranging, we can rewrite equation (9) as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\sum_{k=0}^{n}\sum_{j=0}^{m}\widetilde{a}_j^{(k)}b_{m-j}^{(n-k)}+0.25c_j^{(k)}c_{m-j}^{(n-k)}\right\}(r-\widehat{\theta})^m\eta^n=0.$$

Since the representation of a power series is unique, we conclude that $\{\ldots\}=0$ for all $(n,m)\in \mathbb{N}_0\times\mathbb{N}_0$. This gives a series of equations for the coefficients $a_m^{(n)}$. Solving these equations yields (21). □

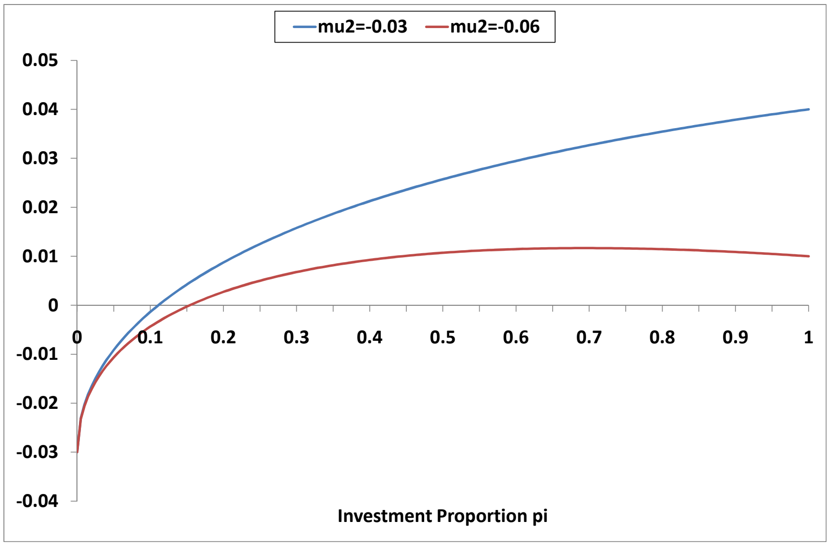

Figure 1: **Functional Forms of the Expected Growth Rate**. The figure illustrates two different forms of the expected growth rate. In both cases, it is assumed that $\mu_0 = -0.03$ and $\mu_1 = 0.1$. For the upper curve, we have $\mu_2 = -0.03$ and for the lower one $\mu_2 = -0.06$.

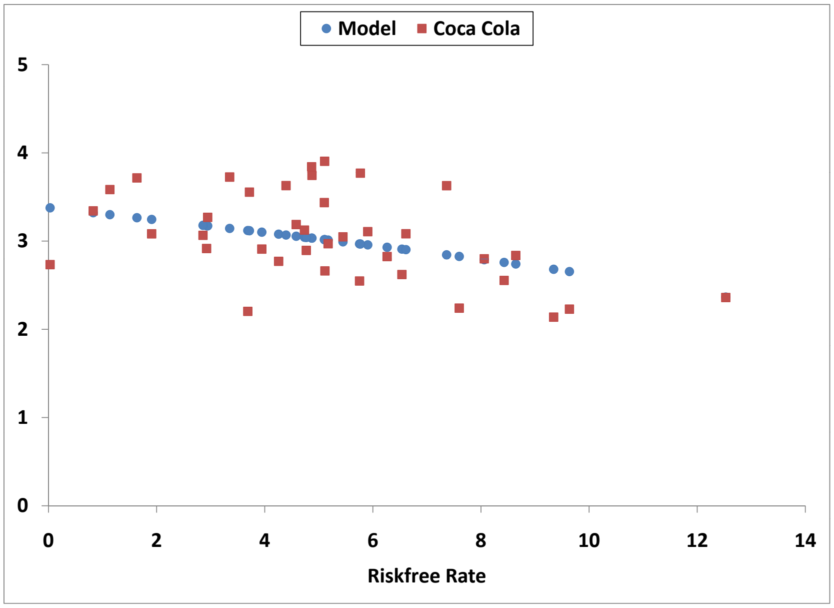

Figure 2: **Log Cash Flow Multiplier of Coca Cola**. This figure depicts the logarithms of 38 observations of Coca Cola's cash flow multiplier over the period from 1971 to 2008, as a function of the riskfree rate. It also shows the fit of our model, which is very close to linear.

	Mean	Std. Dev.	Min.	Max.	Median
Riskfree	5.565	2.856	0.03	16.15	5.107
Real_riskfree	0.912	2.42	-8.184	6.641	1.014
Inflation	4.653	2.992	0.091	14.756	3.645
Slope	1.958	1.421	-2.726	6.511	2.086
Baagov	1.956	0.628	0.754	5.788	1.818
Log_sp_notrend	0.374	0.458	-0.473	1.319	0.369
Vol_sp	14.839	5.575	7.547	40.697	13.426

Table 1: **Summary Statistics for the Macro Variables**. This table provides summary statistics for the macro variables over the period from 1971 to 2008. Riskfree denotes the one-month Fama-French riskfree rate. Inflation denotes the annualized inflation rate calculated from the consumer price index (CPI). Real_riskfree is the difference between both variables. Slope denotes the difference between the 14 year yield on Treasury bonds and the riskfree rate. The maturity of 14 year is chosen since it approximately matches the maturity of the Baa spread as reported by Moody's. Baagov denotes the spread between the yield on Baa corporate bonds and Treasury bonds. Log_sp_notrend denotes the relative value of the S&P 500. This is calculated by taking the logarithm of the index value and subtracting its trend that has been observed from 1926 to 1970. Vol_sp denotes the annualized historical volatility of the S&P 500 calculated using index values of the last 250 trading days.

	Mean	Std. Dev.	Min.	Max.	Median
Log_ratio	2.656	1.078	-5.062	23.827	2.472
Pi	1.021	1.588	0	10.523	0.600
Log_real_size	-0.37	2.234	-12.968	7.927	-0.483
Leverage	0.177	0.172	0	0.952	0.135

Table 2: **Summary Statistics for the Firm Specific Variables**. This table provides summary statistics for the firm specific variables. Log_ratio stands for the logarithm of the cash flow multiplier. Pi denotes the investment proportion given by the ratio between the annual capital expenditures and the free cash flows that is winsorized at the 1% level. Log_real_size denotes the logarithm of the real market capitalization. This variable is inflation adjusted using the consumer price index (CPI). Leverage is the ratio of the debt's book value over the market capitalization of the firm. The statistics are based on 108,443 observations.

	Log_ratio	Real_riskfree	Inflation	Slope	Baagov	Log_sp_notrend	Vol_sp	Pi	Log_real_size	Leverage
Log_ratio	1.000									
Real_riskfree	0.037	1.000								
Inflation	-0.096	-0.477	1.000							
Slope	-0.009	-0.119	-0.262	1.000						
Baagov	-0.076	-0.128	-0.112	0.181	1.000					
Log_sp_notrend	0.111	0.208	-0.674	-0.194	-0.063	1.000				
Vol_sp	-0.067	-0.129	-0.147	0.096	0.708	0.148	1.000			
Pi	0.572	0.026	0.024	-0.011	-0.018	-0.033	-0.024	1.000		
Log_real_size	0.118	-0.039	-0.099	-0.033	-0.009	0.140	-0.026	-0.005	1.000	
Leverage	-0.287	-0.059	0.144	-0.003	0.035	-0.155	0.026	-0.014	-0.092	1.000

Table 3: **Correlation Matrix**. This table reports the correlations between the relevant variables. The correlations are calculated on the basis of 108,443 observations from 1971 to 2008.

	(1)	(2)
Real_riskfree	-0.015*	-0.016
	(-2.43)	(-1.92)
Inflation	-0.020*	-0.020*
	(-2.50)	(-2.39)
Slope	-0.002	-0.003
	(-0.22)	(-0.35)
Baagov	-0.048*	-0.038
	(-2.10)	(-1.71)
Log_sp_notrend	-0.040	0.056
	(-0.86)	(1.05)
Vol_sp	-0.007**	-0.008**
	(-2.85)	(-2.84)
Pi	0.414***	0.399***
	(51.14)	(53.72)
Log_real_size	0.195***	0.067***
	(13.55)	(9.92)
Leverage	-0.619***	-1.296***
	(-13.56)	(-29.02)
Div_dummy	-0.156***	-0.158***
	(-13.89)	(-13.20)
Intercept	2.811***	2.881***
	(32.28)	(27.17)
R^2	0.505	0.478
Fixed effects	yes	no
FF industry dummies	no	yes

Table 4: **Benchmark Regressions**. The table reports the results of panel regressions with Driscoll-Kraay errors that correct for a variety of dependencies including spatial dependencies. The first regression is a pooled regression with fixed effects. The second one is a pooled regression with dummies for the 48 Fama-French industries. All regressions are based on 108,443 observations stemming from 16,567 firms. The time period is 1971 to 2008. The t-statistics are reported in brackets. The significance levels correspond to the following p-values: $*p < 0.05, **p < 0.01, ***p < 0.001$.

	(1)	(3)	(4)	(5)	(6)	(7)
Real_riskfree	-0.015*	-0.014*	-0.014*	-0.004		
	(-2.43)	(-2.06)	(-2.22)	(-0.46)		
Inflation	-0.020*	-0.022*	-0.016	-0.020		
	(-2.50)	(-2.22)	(-1.93)	(-1.58)		
Slope	-0.002	-0.004	-0.001	-0.004		
	(-0.22)	(-0.42)	(-0.06)	(-0.30)		
Baagov	-0.048*	-0.091***		-0.056		
	(-2.10)	(-3.68)		(-1.70)		
Log_sp_notrend	-0.040	-0.070	-0.009	0.011		
	(-0.86)	(-1.24)	(-0.19)	(0.15)		
Vol_sp	-0.007**		-0.011***	-0.013**		
	(-2.85)		(-4.04)	(-3.05)		
Pi	0.414***	0.414***	0.415***		0.414***	0.421***
	(51.14)	(51.56)	(50.89)		(50.48)	(50.45)
Log_real_size	0.195***	0.197***	0.194***		0.203***	
	(13.55)	(13.65)	(13.15)		(18.70)	
Leverage	-0.619***	-0.636***	-0.627***		-0.694***	
	(-13.56)	(-12.81)	(-14.33)		(-14.27)	
Div_dummy	-0.156***	-0.156***	-0.158***		-0.168***	
	(-13.89)	(-13.15)	(-13.90)		(-10.04)	
Intercept	2.811***	2.821***	2.738***	3.055***	2.506***	2.225***
	(32.28)	(27.55)	(30.39)	(22.40)	(103.31)	(96.42)
R^2	0.505	0.504	0.504	0.016	0.499	0.450

Table 5: **Regressions with Excluded Variables.** The table reports the results of panel regressions when some of the explanatory variables are excluded. All regressions are fixed effects regressions with Driscoll-Kraay errors. The first regression corresponds to the first regression that is reported in Table 4. The reported R^2s are the within R^2s of the fixed effect regressions. All regressions are based on 108,443 observations stemming from 16,567 firms. The time period is 1971 to 2008. The t-statistics are reported in brackets. The significance levels correspond to the following p-values: $*p < 0.05, **p < 0.01, ***p < 0.001$.

	(1)	(8)	(9)
Real_riskfree	-0.015*	-0.019**	
	(-2.43)	(-2.67)	
Inflation	-0.020*	-0.023**	
	(-2.50)	(-3.04)	
Slope	-0.002	0.000	
	(-0.22)	(0.02)	
Baagov	-0.048*	-0.041	
	(-2.10)	(-1.89)	
Log_sp_notrend	-0.040	0.012	
	(-0.86)	(0.24)	
Vol_sp	-0.007**	-0.007**	
	(-2.85)	(-2.82)	
Pi	0.414***	0.729***	0.749***
	(51.11)	(23.25)	(22.58)
Pi^2		-0.035***	-0.037***
		(-12.95)	(-12.73)
Log_real_size	0.195***	0.180***	
	(13.57)	(12.73)	
Leverage	-0.619***	-0.596***	
	(-13.53)	(-14.03)	
Div_dummy	-0.156***	-0.148***	
	(-13.89)	(-13.09)	
Intercept	2.811***	2.578***	2.022***
	(32.29)	(32.26)	(75.81)
R^2	0.505	0.538	0.486

Table 6: **Regression with Squared Pi**. The table reports regression results if we include the nonlinear term π^2. Regression (1) is our benchmark regression. The t-statistics are reported in brackets. The significance levels correspond to the following p-values: $*p < 0.05, **p < 0.01, ***p < 0.001$.

	(2)	(10)	(11)
Real_riskfree	-0.016	-0.016**	-0.013***
	(-1.92)	(-3.25)	(-8.75)
Inflation	-0.020*	-0.020**	-0.019***
	(-2.39)	(-3.13)	(-11.39)
Slope	-0.003	-0.003	-0.000
	(-0.35)	(-0.38)	(-0.16)
Baagov	-0.038	-0.038*	-0.044***
	(-1.71)	(-2.31)	(-10.16)
Log_sp_notrend	0.056	0.056	0.007
	(1.05)	(1.59)	(0.69)
Vol_sp	-0.008**	-0.008***	-0.008***
	(-2.84)	(-5.15)	(-15.23)
Pi	0.399***	0.399***	0.411***
	(53.72)	(103.61)	(144.70)
Log_real_size	0.067***	0.067***	0.129***
	(9.92)	(16.89)	(42.98)
Leverage	-1.296***	-1.296***	-0.935***
	(-29.02)	(-32.51)	(-36.49)
Div_dummy	-0.158***	-0.158***	-0.172***
	(-13.20)	(-14.99)	(-19.79)
Intercept	2.881***	2.881***	2.887***
	(27.17)	(34.35)	(44.20)

Table 7: **Robustness Checks for the Standard Errors**. The table reports two additional regressions that we run as robustness checks for the standard errors. The first regression corresponds to the second regression that is reported in Table 4. As in Table 4 and 5, all regressions are based on 108,443 observations stemming from 16,567 firms. The time period is 1971 to 2008. The t-statistics are reported in brackets. The significance levels correspond to the following p-values: $*p < 0.05, **p < 0.01, ***p < 0.001$.

	(1)	(12)	(13)
Real_riskfree	-0.015*	-0.020**	-0.019*
	(-2.43)	(-2.58)	(-2.19)
Inflation	-0.020*	-0.024***	-0.025**
	(-2.50)	(-3.29)	(-3.08)
Slope	-0.002	0.003	0.006
	(-0.22)	(0.41)	(0.66)
Baagov	-0.048*	-0.037	-0.026
	(-2.10)	(-1.68)	(-1.16)
Log_sp_notrend	-0.040	0.050	0.066
	(-0.86)	(1.01)	(1.19)
Vol_sp	-0.007**	-0.007**	-0.009**
	(-2.85)	(-2.82)	(-3.12)
Pi	0.414***		
	(51.11)		
Pi5		0.963***	
		(34.44)	
Pi<1			1.768***
			(28.98)
Log_real_size	0.195***	0.173***	0.179***
	(13.57)	(12.04)	(12.83)
Leverage	-0.619***	-0.556***	-0.390***
	(-13.53)	(-14.28)	(-9.31)
Div_dummy	-0.156***	-0.153***	-0.192***
	(-13.89)	(-10.89)	(-11.78)
Intercept	2.811***	2.394***	2.092***
	(32.29)	(29.76)	(24.31)
R^2	0.5051	0.4835	0.3476
Mean Pi	1.021	0.802	0.5966
Median Pi	0.600	0.600	0.600

Table 8: **Regressions for Different Criteria of Winsorizing Pi**. The table reports regression results if we account for outliers in Pi in different ways. Regression (1) is our benchmark regression where Pi is winsorized at the 1% level. Regression (12) is similar to our benchmark regression, but Pi is winsorized at the 5% level. Regression (13) is similar to our benchmark regression, but Pi is set to one if it is above one. As the benchmark regression (1), all regressions are based on 108,443 observations stemming from 16,567 firms. The t-statistics are reported in brackets. The significance levels correspond to the following p-values: $*p < 0.05, **p < 0.01, ***p < 0.001$.

	(1)	(15)	(16)	(17)
Real_riskfree	-0.015*	-0.010	-0.013*	-0.011
	(-2.43)	(-1.74)	(-2.26)	(-1.63)
Inflation	-0.020*	-0.015	-0.018*	-0.013
	(-2.50)	(-1.74)	(-2.24)	(-1.44)
Slope	-0.002	-0.002	-0.002	-0.003
	(-0.22)	(-0.22)	(-0.20)	(-0.24)
Baagov	-0.048*	-0.050*	-0.049*	-0.063*
	(-2.10)	(-2.22)	(-2.15)	(-2.57)
Log_sp_notrend	-0.040	-0.087	-0.056	-0.092
	(-0.86)	(-1.88)	(-1.21)	(-1.89)
Vol_sp	-0.007**	-0.007**	-0.007**	-0.006*
	(-2.85)	(-2.67)	(-2.76)	(-2.51)
Pi	0.414***		0.365***	0.315***
	(51.11)		(88.90)	(68.50)
Pi_total		0.252***		
		(140.86)		
Pi_rd			0.124***	0.148***
			(22.94)	(32.49)
Log_real_size	0.195***	0.222***	0.205***	0.256***
	(13.57)	(15.05)	(14.12)	(12.78)
Leverage	-0.619***	-0.529***	-0.570***	-0.516***
	(-13.53)	(-11.39)	(-12.66)	(-8.68)
Div_dummy	-0.156***	-0.162***	-0.153***	-0.162***
	(-13.89)	(-12.36)	(-12.83)	(-9.90)
Intercept	2.811***	2.832***	2.787***	2.859***
	(32.29)	(30.37)	(31.42)	(29.38)
R^2	0.505	0.521	0.5338	0.601

Table 9: **Regressions for Different Definitions of Pi**. The table reports regression results if we account for outliers in Pi in different ways. Regression (1) is our benchmark regression where Pi is winsorized at the 1% level. In regression (15), Pi is defined as the ratio between capital expenditures plus R&D expenses (if any) over free cash flows. In regressions (16) and (17), we include an investment fraction for R&D expenses. In regression (16) we set this ratio equal to zero if R&D investments are not reported, whereas in (17) we disregard the observations. Consequently, regression (17) is based on 53,887 observations stemming from 9,291 firms, whereas regressions (1), (15), and (16) are based on 108,443 observations stemming from 16,567 firms. The t-statistics are reported in brackets. The significance levels correspond to the following p-values: $*p < 0.05, **p < 0.01, ***p < 0.001$.

	(18)	(19)	(20)
Real_riskfree	-0.018*	-0.015	-0.014
	(-2.20)	(-1.84)	(-1.85)
Inflation	-0.020*	-0.019*	-0.014
	(-2.40)	(-2.20)	(-1.60)
Slope	-0.003	-0.001	-0.009
	(-0.33)	(-0.14)	(-0.95)
Baagov	-0.049*	-0.046*	-0.057**
	(-2.24)	(-2.01)	(-2.87)
Log_sp_notrend	0.069	0.061	-0.008
	(1.33)	(1.13)	(-0.19)
Vol_sp	-0.007*	-0.008**	-0.008**
	(-2.56)	(-2.81)	(-3.19)
Pi	0.384***	0.393***	0.395***
	(54.34)	(55.29)	(54.96)
Log_real_size	0.056***	0.065***	0.061***
	(7.40)	(8.84)	(8.17)
Leverage	-1.593***	-1.300***	-1.452***
	(-26.97)	(-28.46)	(-26.74)
Div_dummy	-0.160***	-0.132***	-0.147***
	(-11.27)	(-9.97)	(-11.04)
Av_pi		-1.097***	
		(-20.90)	
Av_pi_annual			-0.504***
			(-8.93)
Intercept	2.929***	3.417***	3.189***
	(30.11)	(31.91)	(35.23)
R^2	0.429	0.463	0.445

Table 10: **Regressions Including Average Industry Investments**. The table reports regression results if we include the average πs of the Fama-French industries as additional explanatory variables. These are pooled OLS regressions with Driscoll-Kraay errors where we neither include fixed effects nor Fama-French industry dummies. The variable Av_pi denotes the average π of the corresponding industry over the whole sample period from 1971 to 2008. In contrast, Av_pi_annual denotes the average π of the corresponding industry calculated every year, i.e. these are 48 time series. Regression (18) includes the same explanatory variables as the benchmark regression (1). As the benchmark regression (1), all regressions are based on 108,443 observations stemming from 16,567 firms. The t-statistics are reported in brackets. The significance levels correspond to the following p-values: $*p < 0.05, **p < 0.01, ***p < 0.001$.

Technical Appendix to Cash Flow Multipliers and Optimal Investment Decisions

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C Pricing Kernel Formulation

This part of the Appendix provides an alternative derivation of our results. As in the body of the paper, we consider a model with cash flow dynamics

$$dC = C[\mu(\pi, X)dt + \sigma(\pi, X)dW_c], \quad C(0) = c. \tag{23}$$

For simplicity, we assume that the state process is one-dimensional where the factor is the short rate, $X = r$, possessing the dynamics

$$dr = (\theta - \kappa r)dt + \eta dW_r. \tag{24}$$

We can rewrite the cash flow dynamics in terms of two independent Brownian motions W_r and $\hat{W}_c$:

$$dC = C[\mu(\pi, r)dt + \sigma(\pi, r)(\rho dW_r + \sqrt{1-\rho^2}d\hat{W}_c)], \tag{25}$$

where $< W_r, W_c >_t = \rho t$. Following Ang and Liu (2007b), among others, the pricing kernel (syn. deflator, stochastic discount factor) of the economy is assumed to be of the form

$$d\Lambda = -\Lambda[rdt + \xi_r dW_r + \xi_c d\hat{W}_c], \tag{26}$$

where ξ_r and ξ_c are market prices of risk. Notice that ξ_c can be zero if W_c models idiosyncratic shocks. We can now rewrite the deflator as

$$\Lambda_t = e^{-\int_0^t r_s\, ds} Z_t, \tag{27}$$

where Z is the density of the risk-neutral measure Q and has the dynamics

$$dZ = -Z[\xi_r dW_r + \xi_c d\hat{W}_c]. \tag{28}$$

Notice that under Q

$$dW_r^Q = dW_r + \xi_r dt \quad \text{and} \quad d\hat{W}_c^Q = d\hat{W}_c + \xi_c dt \tag{29}$$

are Brownian increments. Having specified a deflator, the initial firm value is given by

$$V(c, r) = \max_{\pi} \mathrm{E}\Big[\int_0^\infty \Lambda_s(C_s - I_s)ds\Big]. \tag{30}$$

Changing the measure leads to

$$V(c, r) = \max_{\pi} \mathrm{E}^Q\Big[\int_0^\infty e^{-\int_0^s r_u\, du}(1-\pi_s)C_s ds\Big], \tag{31}$$

where we use that $I = \pi C$. The cash flow dynamics under Q are given by

$$dC = C[(\mu(\pi, r) - \sigma(\pi, r)\{\rho\xi_r(r) + \sqrt{1-\rho^2}\xi_c(r)\})dt + \sigma(\pi, r)dW_c^Q], \tag{32}$$

where $dW_c^Q = \rho dW_r^Q + \sqrt{1-\rho^2}d\hat{W}_c^Q$ is a Q-Brownian increment. Recall that in the body of the paper it is assumed that

$$\mu(\pi, r) = \mu_0(r) + \mu_1\sqrt{\pi} + \mu_2\pi, \qquad \sigma(\pi) = \sigma_0 + \sigma_1\sqrt{\pi} + \sigma_2\pi. \tag{33}$$

Now, assume that the market prices of risk are constant[25] such that $\bar{\xi} \equiv \rho\xi_r(r) + \sqrt{1-\rho^2}\xi_c(r) = const$. Then the cash flow dynamics become

$$dC = C[(\mu_0^Q(r) + \mu_1^Q\sqrt{\pi} + \mu_2^Q\pi)dt + \sigma(\pi)dW_c^Q], \tag{34}$$

where $\mu_0^Q(r) \equiv \mu_0(r) - \sigma_0\bar{\xi}$, $\mu_1^Q \equiv \mu_1 - \sigma_1\bar{\xi} = const$, and $\mu_2^Q \equiv \mu_2 - \sigma_2\bar{\xi} = const$. Therefore, the structure of our model is preserved under the risk-neutral measure Q, i.e. we can formulate the HJB etc in terms of the risk-neutral measure. In particular, if $\mu_0(r) = \overline{\mu}_0 + \widehat{\mu}_0 r$, then

$$\mu_0^Q(r) = \overline{\mu}_0^Q + \widehat{\mu}_0 r, \tag{35}$$

where $\overline{\mu}_0^Q \equiv \overline{\mu}_0 - \sigma_0\bar{\xi} = const$.

If interest rates are constant, then $\bar{\xi} = \xi_c$ and $\widehat{\mu}_0 = 0$. We can then apply the result from Theorem 3 of the paper to conclude that the relative option value decreases if $\overline{\mu}_0^Q$ decreases. This can now happen in three ways: Either $\overline{\mu}_0$ decreases or the market price of risk ξ_c or the volatility σ_0 increase (given that both are positive).

D Two State Processes

We consider an economy that is driven by two state processes Y and Z that have Vasicek dynamics

$$\begin{aligned} dY &= (\theta_Y - \kappa_Y Y)dt + \eta_Y dW_Y, \\ dZ &= (\theta_Z - \kappa_Z Z)dt + \eta_Z dW_Z. \end{aligned} \tag{36}$$

The cash flow process of the firm is given by

$$dC = C[\mu(\pi, Y, Z)dt + \sigma(\pi)dW], \quad C(0) = c, \tag{37}$$

[25] In principle, one can also work with market prices of risk that depend on r, but some of the explicitness of the results is then lost.

where

$$\begin{aligned}\mu(\pi, Y, Z) &= \overline{\mu}_0 + \mu_0^Y Y + \mu_0^Z Z + \widehat{\mu}_0 YZ + \mu_1\sqrt{\pi} + \mu_2\pi, \\ \sigma(\pi) &= \sigma(\pi) = \sigma_0 + \sigma_1\sqrt{\pi} + \sigma_2\pi\end{aligned} \tag{38}$$

with constants $\overline{\mu}_0$, μ_0^Y, μ_0^Z, $\widehat{\mu}_0$, μ_1, and μ_2. The processes W, W_Y, and W_Z are correlated Brownian motions with the constant correlations ρ_{YC}, ρ_{ZC}, and ρ_{YZ}. The firm value reads

$$V(c, y, z) = \max_{\pi} \mathrm{E}\Big[\int_0^\infty e^{-\int_0^s R_u\,du}(C_s - I_s)ds\Big], \tag{39}$$

where, with a slight abuse of notation, the risk-adjusted discount rate R is of the form

$$R(Y, Z) = \overline{r} + r^Y Y + r^Z Z + \widehat{r} YZ \tag{40}$$

with constants $\overline{r}$, r^Y, r^Z, and $\widehat{r}$. This specification gives us some flexibility and allows for several possible interpretations. For instance, assume that Y is the default-free interest rate. Then, one could choose R to be of the form

$$R = Y + \beta\lambda, \tag{41}$$

where β is the firm's beta and λ is the risk premium. If the default-free interest rate predicts the risk premium, then one can set

$$\lambda = \overline{\lambda} + \lambda^Y Y. \tag{42}$$

Then Z could model a stochastic beta of the firm. In this case,

$$R = Y + Z(\overline{\lambda} + \lambda^Y Y) = Y + \overline{\lambda} Z + \lambda^Y YZ \tag{43}$$

or in our above notation

$$\overline{r} = 0, \quad r^Y = 1, \quad r^Z = \overline{\lambda}, \quad \widehat{r} = \lambda^Y. \tag{44}$$

Alternatively, one could assume the beta of the firm to be constant and identify Z with the risk premium. Then,

$$R = Y + \beta Z, \tag{45}$$

or in our notation above

$$\overline{r} = 0, \quad r^Y = 1, \quad r^Z = \beta = const, \quad \widehat{r} = 0. \tag{46}$$

The Bellman equation for this problem reads

$$\begin{aligned}0 &= \max_{\pi}\Big\{\mu(y,z,\pi)cV_c + 0.5\sigma^2(\pi)cV_{cc} - R(y,z)V + c - \pi c + (\theta_Y - \kappa_Y y)V_y + 0.5\eta_Y^2 V_{yy} \\ &\quad +(\theta_Z - \kappa_Z z)V_z + 0.5\eta_Z^2 V_{zz} + \eta_Y\eta_Z\rho_{YZ}V_{yz} + \eta_Y\sigma(\pi)c\rho_{YC}V_{yc} + \eta_Z\sigma(\pi)c\rho_{ZC}V_{zc}\Big\}. \qquad (47)\end{aligned}$$

We conjecture the following form of the firm value

$$V(c,y,z) = cf(y,z) \qquad (48)$$

and obtain

$$\begin{aligned}0 &= \max_{\pi}\Big\{\mu(y,z,\pi)f - R(y,z)f + 1 - \pi + (\theta_Y - \kappa_Y y)f_y + 0.5\eta_Y^2 f_{yy} \qquad (49)\\ &\quad +(\theta_Z - \kappa_Z z)f_z + 0.5\eta_Z^2 f_{zz} + \eta_Y\eta_Z\rho_{YZ}f_{yz} + \eta_Y\sigma(\pi)\rho_{YC}f_y + \eta_Z\sigma(\pi)\rho_{ZC}f_z\Big\}.\end{aligned}$$

Notice that the term involving V_{cc} drops out since the firm value is linear in the current cash flow. The first-order condition for the optimal investment proportion reads

$$\pi^* = \left(\frac{\mu_1 f + \eta_Y\sigma_1\rho_{YC}f_y + \eta_Z\sigma_1\rho_{ZC}f_z}{2(1-\mu_2 f - \eta_Y\sigma_2\rho_{YC}f_y - \eta_Z\sigma_2\rho_{ZC}f_z)}\right)^2. \qquad (50)$$

Substituting back into the Bellman equation leads to a non-linear second-order partial differential equation for f:

$$\begin{aligned}0 &= (\overline{\alpha} + \alpha_Y y + \alpha_Z z + \widehat{\alpha}yz)f + 1 + (\theta_Y + \eta_Y\sigma_0\rho_{YC} - \kappa_Y y)f_y + 0.5\eta_Y^2 f_{yy} \qquad (51)\\ &\quad +(\theta_Z + \eta_Z\sigma_0\rho_{ZC} - \kappa_Z z)f_z + 0.5\eta_Z^2 f_{zz} + \eta_Y\eta_Z\rho_{YZ}f_{yz} \\ &\quad +0.25\frac{(\mu_1 f + \eta_Y\sigma_1\rho_{YC}f_y + \eta_Z\sigma_1\rho_{ZC}f_z)^2}{1-\mu_2 f - \eta_Y\sigma_2\rho_{YC}f_y - \eta_Z\sigma_2\rho_{ZC}f_z},\end{aligned}$$

where $\overline{\alpha} = \overline{\mu}_0 - \overline{r}$, $\alpha_Y = \mu_0^Y - r^Y$, $\alpha_Z = \mu_0^Z - r^Z$, and $\widehat{\alpha} = \widehat{\mu}_0 - \widehat{r}$. We solve this equation in two steps. Firstly, we expand f in terms of η_Y and η_Z in the following way

$$f(y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A^{n,m}(y,z)(\eta_Y)^n(\eta_Z)^m. \qquad (52)$$

This leads to the following result.

Proposition 5 (PDEs for $A^{n,m}$)**.** *The functions $A^{n,m}$ satisfy the following series of partial differential equations*

$$\sum_{p=0}^{i}\sum_{q=0}^{k}\Big\{\widetilde{A}^{p,q}(y,z)B^{i-p,k-q}(y,z) + 0.25C^{p,q}(y,z)C^{i-p,k-q}(y,z)\Big\} = 0, \quad (i,k)\in I\!N_0 \times I\!N_0, \qquad (53)$$

where

$$\begin{aligned}
\widetilde{A}^{n,m} &= \mathbf{1}_{\{n=m=0\}} - \mu_2 A^{n,m} - \sigma_2\rho_{YC}A_y^{n-1,m} - \sigma_2\rho_{ZC}A_z^{n,m-1}, \qquad (54)\\
B^{n,m} &= \mathbf{1}_{\{n=m=0\}} + (\overline{\alpha} + \alpha_Y y + \alpha_Z z + \widehat{\alpha} yz)A^{n,m} + (\theta_Y - \kappa_Y y)A_y^{n,m} + \sigma_0\rho_{YC}A_y^{n-1,m}\\
&\quad +0.5A_{yy}^{n-2,m} + (\theta_Z - \kappa_Z z)A_z^{n,m} + \sigma_0\rho_{ZC}A_z^{n,m-1} + 0.5A_{zz}^{n,m-2} + \rho_{YZ}A_{yz}^{n-1,m-1},\\
C^{n,m} &= \mu_1 A^{n,m} + \sigma_1\rho_{YC}A_y^{n-1,m} + \sigma_1\rho_{ZC}A_z^{n,m-1},
\end{aligned}$$

with the convention that coefficients with negative indices are zero.

Proof. Substituting (52) into (51) and long calculations yield (53). □

We now expand $A^{n,m}$ in terms of the state variables y and z centered at the mean reversion levels $\widehat{\theta}_Y = \theta_Y/\kappa_Y$ and $\widehat{\theta}_Z = \theta_Z/\kappa_Z$, i.e.

$$A^{n,m}(y,z) = \sum_{\nu=0}^{\infty}\sum_{\ell=0}^{\infty} a_{\nu,\ell}^{n,m}\left(y-\widehat{\theta}_Y\right)^{\nu}\left(z-\widehat{\theta}_Z\right)^{\ell}. \qquad (55)$$

This leads to the following representation of f:

$$\begin{aligned}
f(y,z) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A^{n,m}(y,z)(\eta_Y)^n(\eta_Z)^m \qquad (56)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\ell=0}^{\infty} a_{\nu,\ell}^{n,m}\left(y-\widehat{\theta}_Y\right)^{\nu}\left(z-\widehat{\theta}_Z\right)^{\ell}(\eta_Y)^n(\eta_Z)^m\\
&= \sum_{\nu=0}^{\infty}\sum_{\ell=0}^{\infty}\underbrace{\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(\eta_Y)^n(\eta_Z)^m a_{\nu,\ell}^{n,m}\right)}_{=\widehat{a}_{\nu,\ell}}\left(y-\widehat{\theta}_Y\right)^{\nu}\left(z-\widehat{\theta}_Z\right)^{\ell}
\end{aligned}$$

To derive this representation, the following lemma firstly provides the expansions of the functions $\widetilde{A}^{n,m}$, $B^{n,m}$, and $C^{n,m}$.

Lemma 6. *For $(n,m) \in I\!N_0 \times I\!N_0$ we obtain*

$$\begin{aligned}
\widetilde{A}^{n,m} &= \sum_{\nu=0}^{\infty}\sum_{\ell=0}^{\infty}\widetilde{a}_{\nu,\ell}^{n,m}\left(y-\widehat{\theta}_Y\right)^{\nu}\left(z-\widehat{\theta}_Z\right)^{\ell}, \quad B^{n,m} = \sum_{\nu=0}^{\infty}\sum_{\ell=0}^{\infty} b_{\nu,\ell}^{n,m}\left(y-\widehat{\theta}_Y\right)^{\nu}\left(z-\widehat{\theta}_Z\right)^{\ell},\\
C^{n,m} &= \sum_{\nu=0}^{\infty}\sum_{\ell=0}^{\infty} c_{\nu,\ell}^{n,m}\left(y-\widehat{\theta}_Y\right)^{\nu}\left(z-\widehat{\theta}_Z\right)^{\ell}. \qquad (57)
\end{aligned}$$

with[26]

$$\begin{aligned}
\widetilde{a}_{\nu,\ell}^{n,m} &= \mathbf{1}_{\{n,m,\nu,\ell=0\}} - \mu_2 a_{\nu,\ell}^{n,m} - \sigma_2\rho_{YC}(\nu+1)a_{\nu+1,\ell}^{n-1,m} - \sigma_2\rho_{ZC}(\ell+1)a_{\nu,\ell+1}^{n,m-1}, \qquad (58)\\
b_{\nu,\ell}^{n,m} &= \mathbf{1}_{\{n,m,\nu,\ell=0\}} + (\overline{\overline{\alpha}} - \nu\kappa_Y - \ell\kappa_Z)a_{\nu,\ell}^{n,m} + \widehat{b}_{\nu,\ell}^{n,m},\\
c_{\nu,\ell}^{n,m} &= \mu_1 a_{\nu,\ell}^{n,m} + \sigma_1\rho_{YC}(\nu+1)a_{\nu+1,\ell}^{n-1,m} + \sigma_1\rho_{ZC}(\ell+1)a_{\nu,\ell+1}^{n,m-1},
\end{aligned}$$

[26] $\mathbf{1}_{\{n,m,\nu,\ell=0\}}$ is one if $n = m = \nu = \ell = 0$ and zero otherwise.

where

$$\overline{\overline{\alpha}} = \overline{\alpha} + \alpha_Y \widehat{\theta}_Y + \alpha_Z \widehat{\theta}_Z + \widehat{\alpha}\widehat{\theta}_Y \widehat{\theta}_Z, \tag{59}$$

$$\begin{aligned} \widehat{b}^{n,m}_{\nu,\ell} &= (\alpha_Y + \widehat{\alpha}\widehat{\theta}_Z) a^{n,m}_{\nu-1,\ell} + (\alpha_Z + \widehat{\alpha}\widehat{\theta}_Y) a^{n,m}_{\nu,\ell-1} + \widehat{\alpha} a^{n,m}_{\nu-1,\ell-1} + \sigma_0 \rho_{YC}(\nu+1) a^{n-1,m}_{\nu+1,\ell} \\ &\quad + 0.5(\nu+2)(\nu+1) a^{n-2,m}_{\nu+2,\ell} + \sigma_0 \rho_{ZC}(\ell+1) a^{n,m-1}_{\nu,\ell+1} + 0.5(\ell+2)(\ell+1) a^{n,m-2}_{\nu,\ell+2} \\ &\quad + \rho_{YZ}(\nu+1)(\ell+1) a^{n-1,m-1}_{\nu+1,\ell+1} \end{aligned}$$

Remark. Splitting up $b^{n,m}_{\nu,\ell}$ into a term involving $a^{n,m}_{\nu,\ell}$ and into the term $\widehat{b}^{n,m}_{\nu,\ell}$ will be useful later on. This is because $\widehat{b}^{n,m}_{\nu,\ell}$ involves lower order coefficients only that are known when one calculates $a^{n,m}_{\nu,\ell}$ with the help of a recursion that we will provide below.

Combining our results above, we can rewrite the Bellman equation (51) in the following way:

$$0 = \sum_{\nu=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \Lambda^{N,M}_{\nu,\ell} \left(y - \widehat{\theta}_Y\right)^{\nu} \left(z - \widehat{\theta}_Z\right)^{\ell} (\eta_Y)^N (\eta_Z)^M, \tag{60}$$

where

$$\Lambda^{N,M}_{\nu,\ell} = \sum_{p=0}^{\nu} \sum_{q=0}^{\ell} \sum_{n=0}^{N} \sum_{m=0}^{M} \underbrace{\widetilde{a}^{n,m}_{p,q} b^{N-n,M-m}_{\nu-p,\ell-q} + 0.25 c^{n,m}_{p,q} c^{N-n,M-m}_{\nu-p,\ell-q}}_{=H^{n,m,N-m,M-m}_{p,q,\nu-p,\ell-q}}. \tag{61}$$

Since the representation of a power series is unique, we obtain that for all combinations $(\nu, \ell, N, M) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$

$$\Lambda^{N,M}_{\nu,\ell} = 0. \tag{62}$$

We thus obtain the following result.

Proposition 7 (Recursion for $a^{N,M}_{\nu,\ell}$). *The coefficients are given by the following recursion*

$$\begin{aligned} a^{N,M}_{0,0} &= -\frac{\sum_{(n,m)\in\mathcal{K}} H^{n,m,N-m,M-m}_{0,0,0,0} + R^{N,M}_{0,0}}{D_{0,0}}, \\ a^{N,M}_{\nu,\ell} &= -\frac{\sum_{(p,q,n,m)\in\mathcal{I}} H^{n,m,N-m,M-m}_{p,q,\nu-p,\ell-q} + R^{N,M}_{\nu,\ell}}{D_{\nu,\ell}}, \end{aligned} \tag{63}$$

where

$$\begin{aligned} R^{N,M}_{\nu,\ell} &= (1 - \mu_2 a^{0,0}_{0,0}) \left[\mathbf{1}_{\{N,M,\nu,\ell=0\}} + \widehat{b}^{n,m}_{\nu,\ell}\right] \\ &\quad + (1 + \overline{\overline{\alpha}} a^{0,0}_{0,0}) \left[\mathbf{1}_{\{N,M,\nu,\ell=0\}} - \sigma_2 \rho_{YC}(\nu+1) a^{N-1,M}_{\nu+1,\ell} - \sigma_2 \rho_{ZC}(\ell+1) a^{N,M-1}_{\nu,\ell+1}\right] \\ &\quad + 0.5 \mu_1 \sigma_1 a^{0,0}_{0,0} \left[\rho_{YC}(\nu+1) a^{N-1,M}_{\nu+1,\ell} + \rho_{ZC}(\ell+1) a^{N,M-1}_{\nu,\ell+1}\right], \\ D_{\nu,\ell} &= (1 - \mu_2 a^{0,0}_{0,0})(\overline{\overline{\alpha}} - \nu\kappa_Y - \ell\kappa_Z) - \mu_2(1 + \overline{\overline{\alpha}} a^{0,0}_{0,0}) + 0.5 \mu_1^2 a^{0,0}_{0,0} \end{aligned} \tag{64}$$

and $\mathcal{K} = \{0,1,\dots,N-1,N\} \times \{0,1,\dots,M-1,M\} \setminus \{(0,0),(N,M)\}$ *as well as* $\mathcal{J} = \{0,1,\dots,\nu-1,\nu\} \times \{0,1,\dots,\ell-1,\ell\} \times \{0,1,\dots,N-1,N\} \times \{0,1,\dots,M-1,M\} \setminus \{(0,0,0,0),(\nu,\ell,N,M)\}$ *are index sets.*[27]

We emphasize that this recursion is explicit. Although the previous proposition is also valid for $a^{0,0}_{\nu,\ell}$, we summarize the corresponding results in a separate corollary. In particular, the equation for $a^{0,0}_{0,0}$ is special because in this case $a^{0,0}_{0,0}$ appears on both sides of equation (63). This is the only equation of the recursion that is non-linear.

Corollary 8 (Representation of $A^{0,0}$)**.** *The coefficient* $a^{0,0}_{0,0}$ *satisfies the quadratic equation*

$$0 = (0.25\mu_1^2 - \mu_2\overline{\overline{\alpha}})(a^{0,0}_{0,0})^2 + (\overline{\overline{\alpha}} - \mu_2)a^{0,0}_{0,0} + 1. \tag{65}$$

The subsequent coefficients can be calculated from the explicit representation

$$a^{0,0}_{\nu,\ell} = -\frac{\sum_{(p,q)\in\mathcal{J}} H^{0,0,0,0}_{p,q,\nu-p,\ell-q} + R^{0,0}_{\nu,\ell}}{D_{\nu,\ell}} \tag{66}$$

where

$$\begin{aligned} R^{0,0}_{\nu,\ell} \quad = \quad & (1-\mu_2 a^{0,0}_{0,0})\left[\mathbf{1}_{\{\nu,\ell=0\}} + (\alpha_Y + \widehat{\alpha}\widehat{\theta}_Z)a^{0,0}_{\nu-1,\ell} + (\alpha_Z + \widehat{\alpha}\widehat{\theta}_Y)a^{0,0}_{\nu,\ell-1} + \widehat{\alpha}a^{0,0}_{\nu-1,\ell-1}\right] \\ & + (1+\overline{\overline{\alpha}}a^{0,0}_{0,0})\mathbf{1}_{\{\nu,\ell=0\}} \end{aligned} \tag{67}$$

and $\mathcal{J} = \{0,1,\dots,\nu-1,\nu\} \times \{0,1,\dots,\ell-1,\ell\} \setminus \{(0,0),(\nu,\ell)\}$ *is an index set.*

Notice that (65) becomes (16) if the state processes are constant.

[27] Therefore, the sum $\sum_{n=0}^{N}\sum_{m=0}^{M}\dots$ has two more elements than $\sum_{(n,m)\in\mathcal{K}}\dots$, namely the elements with indices $(n,m)=(0,0)$ and $(n,m)=(M,N)$. The same property holds for the index set $\mathcal{I}$ accordingly.