Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing

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Abstract

We study the implications of preference heterogeneity for asset pricing. We use recursive preferences in order to separate heterogeneity in risk aversion from heterogeneity in the intertemporal elasticity of substitution, and an overlapping-generations framework to obtain a non-degenerate stationary equilibrium. We solve the model explicitly up to the solutions of ordinary differential equations, and highlight the effects of overlapping generations and each dimension of preference heterogeneity on the market price of risk, interest rates, and the volatility of stock returns.

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Macro-finance models routinely postulate economies populated by a single, price-taking, "representative agent" with constant relative risk aversion and intertemporal elasticity of substitution (IES). A legitimate question is how preference heterogeneity, a widely documented and intuitively plausible feature of reality, would change the conclusions of such a parsimonious setup.

With a few exceptions,[1] the literature on preference heterogeneity has addressed this question with models featuring infinitely-lived agents who maximize expected-utility, constant-relative-risk-aversion (CRRA) preferences. Such models imply generically that one type of agents dominates the consumption and wealth distributions in the long run, because preference heterogeneity translates into cross-sectional differences in average growth rates of wealth. In turn, the ensuing degenerate cross-sectional distribution for consumption and wealth implies that all asset-pricing quantities (price-dividend ratios, equity premia, etc.) converge to the constant levels obtaining when the economy is populated by only one type of agent. As a result, it becomes difficult to compare these models to a large body of empirical literature predicated on a non-degenerate stationary distribution for these quantities.[2] Furthermore, CRRA preferences tie a consumer's IES to her risk aversion, making it impossible to separate heterogeneity along these two conceptually different preference characteristics.

To address these two limitations, we study an economy populated by two agent types with heterogeneous, recursive preferences (Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989)). This feature allows us to separate the risk aversion and the IES of the agents. Agents have finite and stochastic lifetimes, as in the continuous-time, overlapping-generations (OLG) economy of Blanchard (1985). Due to finite lifetimes, no group of agents accumulates wealth indefinitely. Furthermore, all newly born agents have the same endowments, regardless of their preferences. Therefore, despite the cross-sectional differences in average growth rates of wealth, no type of agent dominates the economy in the long run,

[1] We postpone a detailed discussion of the literature for the latter part of the introduction.

[2] To provide an example, the empirical literature on return predictability considers regressions of excess returns on a constant and various predictors, such as the price-to-dividend ratio. The identifying assumption of ordinary least squares (in large samples) requires that the regressors have non-degenerate asymptotic distributions, i.e., do not converge to constants (otherwise, the variance-covariance matrix is singular).

and stationary distributions are non-degenerate.

Our asset-pricing findings can be grouped in three broad categories, concerning (i) the market price of risk, (ii) the interest rate, and (iii) the return volatility and equity premium. We discuss each of them in turn.

The market price of risk, defined as the ratio of stock-market excess return to stock-market volatility, is a useful summary statistic of the risk-return tradeoff in an economy. We show that, in the special case of CRRA utilities, our OLG structure preserves a familiar result in the literature: The market price of risk is a countercyclical, weighted average of the market prices of risk in two representative-agent economies, populated exclusively by the more risk-averse, respectively the less risk-averse, agents in our model. This weighted average depends only on the risk aversions and the consumption shares of the two types of agents.

Novel results obtain when agents have heterogeneous recursive preferences. The optimal consumption-allocation rule endogenously generates persistence in individual agents' consumption growth, even if aggregate consumption growth is i.i.d.. Except in the special CRRA case, agents' marginal rates of substitution are affected by such persistence, and so is the market price of risk. We provide conditions under which the resulting market price of risk exceeds the one under CRRA preferences (keeping risk-aversion coefficients fixed). Interestingly, in some illustrative examples the market price of risk exceeds even the value obtaining in the economy populated exclusively by the agent with the higher risk-aversion coefficient.

Our second set of results concerns interest rates. We show that even if all agents have homogeneous preferences, interest rates in an OLG economy are typically lower than in the respective economy populated by infinitely lived agents. The reason is that finitely-lived agents faced with realistic life-cycle earnings profiles need to save in order to provide for themselves in old age. These increased savings lower the interest rate and help address the low risk-free rate puzzle documented by Weil (1989).

Preference heterogeneity generates interest-rate time variation that reflects changes in the consumption shares of different types of agents. Contrary to models where agents have identical recursive preferences, the time variation in interest rates can be unrelated to changes in expected aggregate consumption growth. We argue that this feature has an important consequence for empirical research: If an econometrician were to impose the assumption of a representative agent with the usual recursive preferences and estimate her IES using aggregate data, then the obtained estimate would be biased towards zero. (In the special case of i.i.d. consumption growth the IES estimate would be literally zero.) This observation is consistent with the fact that estimates of the IES based on microeconomic data are typically higher than those based on aggregate consumption data.

Our third set of results concern the volatility of stock-market returns and the equity premium. We first show that, if agents have different IES, but the same risk-aversion coefficient, then the volatility of the stock market and the equity premium (as well as the market price of risk) are constant and equal to their respective values in an economy populated by a single agent with the same risk aversion. In our model, therefore, risk-aversion heterogeneity is essential for addressing certain empirical asset-pricing properties.

However, for a given degree of risk-aversion heterogeneity, heterogeneity in the IES can affect the results significantly. As part of our analysis we discuss in detail the implications and tradeoffs of different joint distributions of the risk aversion and the IES on the cyclicality of interest rates, discount rates, volatility, and the equity premium.

Our paper relates primarily to the analytical asset-pricing literature on preference heterogeneity.[3] As already mentioned, this literature does not separate IES heterogeneity from risk-aversion heterogeneity. Furthermore, with the notable exception[4] of Chan and Kogan (2002), these models imply a generic degeneracy of stationary distributions. The key assumption that ensures stationarity in Chan and Kogan (2002) is that an agent's utility derives exclusively from her consumption relative to aggregate consumption. We use an OLG approach to obtain stationarity (see, e.g.,[5] Spear and Srivastava (1986)), which allows us to

[3]Contributions include Dumas (1989), Wang (1996), Chan and Kogan (2002), Bhamra and Uppal (2009), Longstaff and Wang (2008), and Zapatero and Xiouros (2010).

[4]See also Zapatero and Xiouros (2010) for a discrete-time version of Chan and Kogan (2002).

[5]Spear and Srivastava (1986) discusses the existence of stationary Markov equilibria in OLG models.

study standard CRRA and recursive utility specifications.

A theoretical literature has considered preference aggregation for infinitely-lived agents with heterogeneous, recursive preferences. Early contributions studied deterministic environments.[6] Subsequently, other authors investigated the existence of equilibrium and recursive algorithms for the construction of a representative agent in stochastic environments.[7] The novel aspect of our paper is that we provide analytic expressions for the equilibrium prices and quantities up to the solution of ordinary differential equations. Besides allowing an accurate and efficient computation of the equilibrium, these expressions make it possible to characterize the new equilibrium properties associated with recursive preferences (as compared with CRRA preferences).[8]

Some of our model's channels are reminiscent of leading representative-agent models. Similar to Campbell and Cochrane (1999), the market price of risk is countercyclical. In Campbell and Cochrane (1999) this countercyclicality is due to the assumed countercyclicality of the risk aversion of the representative agent, while in our model it arises as a result of endogenous, countercyclical redistribution of wealth from the less to the more risk averse agents.[9] Also, similar to Bansal and Yaron (2004), persistent consumption growth coupled with recursive preferences generate a higher market price of risk. However, in contrast to Bansal and Yaron (2004), this persistence is the endogenous result of the consumption-allocation rule and it pertains to the consumption growth of *individual agents*. It obtains even if *aggregate* consumption growth is i.i.d. Additionally, our qualitative results do not require an IES above unity for any agent.

However, it focuses on heterogeneous goods (rather than preferences), and the Markov state space contains past prices rather than the consumption distribution.

[6]See, e.g., Lucas and Stokey (1984), Epstein (1987).

[7]See, e.g., Ma (1993), Kan (1995), Duffie et al. (1994)

[8]Backus et al. (2008) give a closed-form solution for the special case where one type of agents is risk neutral and has zero IES, whereas the other type of agents is infinitely risk averse and has infinite IES.

[9]This result was first shown for CRRA utilities by Dumas (1989), and emphasized by Chan and Kogan (2002). We also note that a large asset-pricing literature obtains time-varying market prices of risk as a result of the coexistence of multiple goods or production factors. Indicative examples include Piazzesi et al. (2007), Santos and Veronesi (2006), Tuzel (2010), Papanikolaou (2010), and Gomes et al. (2009). Our model differs in that the result is not due to assumptions on relative endowments of different goods and production factors, but to the interaction of agents with different preferences.

Another relevant literature employs numerical methods to solve life-cycle-of-earnings models in general equilibrium. The paper most closely related to ours is Gomes and Michaelides (2008).[10] Gomes and Michaelides (2008) obtains joint implications for asset returns and stock-market participation decisions in a rich setup that includes costly participation, heterogeneity in both preferences and income, and realistic life cycles. This paper also assumes, however, that the volatility of output, the volatility of stock market returns, and the equity premium are driven by exogenous, random capital-depreciation shocks, rather than the commonly assumed total factor productivity shocks. This assumption implies that stock-market volatility is due entirely to exogenous variation in the quantity of capital, rather than to fluctuations in the price of capital (Tobin's q), in contrast to the data where most of the return volatility is due to fluctuations in the price of capital. This exogenous source of return volatility is essential for their model to produce a non-negligible equity premium.[11]

We consider exclusively endowment shocks in a Lucas (1978)-style economy. While this more conventional asset-pricing framework abstracts from modeling investment, it has the advantage that stock market fluctuations are due to endogenous variations in the price of capital. It also allows us to readily compare our results to the large asset-pricing literature using such a framework. More importantly, endogenizing the price of capital generates new insights compared to Gomes and Michaelides (2008). For instance, as we show in Section 4.3, there is typically a tradeoff between a high market price of risk and a high volatility of returns, which can only be addressed when volatility is driven by endogenous price-of-capital fluctuations.

Another difference with the calibration-oriented literature is the analytical tractability of our framework. Therefore, our work is complementary to quantitative exercises that feature richer setups, but must sacrifice partly the transparency of the mechanisms involved.

We also relate to the literature that analyzes the applications of OLG frameworks to

[10] Other related work includes Storesletten et al. (2007) and Guvenen (2009). Storesletten et al. (2007) study an OLG economy with homogeneous preferences and uninsurable income shocks. Similar to the OLG model of Gomes and Michaelides (2008), Guvenen (2009) studies preference heterogeneity and limited market participation, but his model features inifinitely lived agents.

[11] As Gomes and Michaelides (2008) acknowledge, "Without those shocks, our economy would still have a high market price of risk, but a negligible equity premium".

asset pricing.[12] Many of these models combine the OLG structure with other frictions or shocks to drive incomplete risk-sharing across generations, so that consumption risk is disproportionate for cohorts predominantly participating in asset markets. Even though we think these channels important for asset pricing, we do not include them in order to isolate the intuitions pertaining to preference heterogeneity. Another source of difference is that we model births and deaths in continuous time, similar to Blanchard (1985). As a result, our model produces implications for returns over any duration (month, year, etc.), and not just over the lifespan of a generation.[13]

The paper is structured as follows. Section 1 presents the model. Section 2 discusses the solution obtaining with preference homogeneity, which allows us to isolate the effects of overlapping generations. Section 3 introduces preference heterogeneity, but restricts preferences to the CRRA class. This helps us relate our model to the heterogenous-preferences literature that uses infinitely-lived CRRA agents, and in particular it allows us to illustrate how the OLG feature leads to stationarity. Finally, Section 4 studies recursive preferences. Here we develop some key intuitions in the context of a simple three-period model and then generalize them in the context of the full model. Section 5 concludes. All proofs are in the appendix.

1 Model

1.1 Demographics and preferences

Our specification of demographics follows Blanchard (1985) and Yaari (1965). Time is continuous. Each agent faces a constant hazard rate of death $\pi > 0$ throughout her life, so that a fraction π of the population perishes per unit of time. Simultaneously, a cohort of mass π is born per unit of time. Given these assumptions, the time-t size of a cohort of

[12] Examples of such papers are Abel (2003), Storesletten et al. (2007), Constantinides et al. (2002), Farmer (2002), Piazzesi and Schneider (2009), Heaton and Lucas (2000), and Gârleanu et al. (2009).

[13] Farmer (2002) analyzes a stochastic version of Blanchard (1985) in discrete time, but all agents maximize the same logarithmic preferences.

agents born at some time $s < t$ is given by $\pi e^{-\pi(t-s)}ds$, and the total population size is $\int_{-\infty}^{t} \pi e^{-\pi(t-s)}ds = 1$.[14]

To allow for the separation of the effects of the IES and the risk aversion, we assume that agents have the type of recursive preferences proposed by Epstein and Zin (1989) and Weil (1989) in discrete-time settings and extended by Duffie and Epstein (1992) to continuous-time settings. Specifically, agents maximize

$$V_s \;=\; \mathrm{E}_s\left[\int_s^{\infty} f\left(c_u, V_u\right) du\right], \tag{1}$$

where $f\left(c, V\right)$ is given by

$$f\left(c, V\right) \;\equiv\; \frac{1}{\alpha}\left(\frac{c^{\alpha}}{\left(\left(1-\gamma\right)V\right)^{\frac{\alpha}{1-\gamma}-1}} - \left(\rho+\pi\right)\left(1-\gamma\right)V\right). \tag{2}$$

The function $f\left(c, V\right)$ aggregates the utility arising from current consumption c and the value function V. The parameter $\gamma > 0$ controls the risk aversion of the agent, while $\left(1-\alpha\right)^{-1}$ gives the agent's IES. We assume that $\alpha < 1$, so that the IES ranges between zero and infinity. The parameter $\rho > 0$ is the agent's subjective discount factor. The online appendix[15] gives a short derivation of the objective function (1) as the continuous-time limit of a discrete-time, recursive-preference specification with random times of death.

In this section and the following one we assume that all agents have the same preferences, to isolate the asset-pricing implications of overlapping generations. Section 3 introduces heterogeneity in preferences.

1.2 Endowments, earnings, and dividends

Aggregate output in the economy is given by

$$\frac{dY_t}{Y_t} \;=\; \mu_Y dt + \sigma_Y dB_t, \tag{3}$$

[14]We assume no population growth for simplicity. Introducing population growth is a straightforward extension.

[15]Available at http://faculty.chicagobooth.edu/stavros.panageas/research/OLGextapp.pdf.

where μ_Y is the growth rate, σ_Y the volatility of output, and B_t a standard Brownian motion.[16]

At time t, an agent born at time s is endowed with earnings $y_{t,s}$, where

$$y_{t,s} \; = \; \omega Y_t \left[\overline{h} G\left(t-s\right) \right], \quad \omega \in (0,1). \tag{4}$$

$G\left(t-s\right) \geq 0$ is a function of age that controls the life-cycle earnings profile. Aggregate earnings are therefore given by

$$\int_{-\infty}^{t} \pi e^{-\pi(t-s)} y_{t,s} ds \; = \; \omega Y_t \overline{h} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} G\left(t-s\right) ds \; = \; \omega Y_t \overline{h} \int_{0}^{\infty} \pi e^{-\pi u} G\left(u\right) du, \tag{5}$$

where the last equality follows by the change of variables $u = t - s$. We assume throughout that $\int_0^\infty \pi e^{-\pi u} G\left(u\right) du < \infty$, and normalize $\overline{h}$ to

$$\overline{h} \; \equiv \; \left(\int_0^\infty \pi e^{-\pi u} G\left(u\right) du \right)^{-1}, \tag{6}$$

so that aggregate earnings are given by ωY_t. The remaining fraction $1-\omega$ of output Y_t is paid out as dividends $D_t \equiv \left(1-\omega\right) Y_t$ by the representative firm. Following Lucas (1978), we assume that the representative firm is a “Lucas tree”, i.e., it simply pays dividends without facing any economic decisions.

1.3 Markets and budget constraints

Agents can allocate their portfolios between shares of the representative firm and instantaneously maturing riskless bonds, which pay a riskless interest rate r_t per dollar invested. The supply of shares of the firm is normalized to one, while bonds are in zero net supply. The price S_t of each share evolves according to

$$dS_t \; = \; \left(\mu_t S_t - D_t\right) dt + \sigma_t S_t dB_t, \tag{7}$$

[16] We fix throughout a probability space and a filtration generated by B satisfying the usual conditions.

where the coefficients μ_t and σ_t, as well as the interest rate r_t, are determined in equilibrium.

Finally, agents can access a market for annuities through competitive insurance companies as in Blanchard (1985). Specifically, an agent born at time s who owns financial wealth equal to $W_{t,s}$ at time t can enter a contract with an annuity company that entitles the agent to receive an income stream of $\pi W_{t,s}$ per unit of time. In exchange, the insurance company collects the agent's financial wealth when she dies. Entering such an annuity contract is optimal for all living agents, given that they have no bequest motives. The law of large numbers implies that insurance companies collect πW_t per unit of time from perishing agents, where W_t denotes aggregate wealth. This allows them to pay an income flow of πW_t to survivors and break even.

Letting $\theta_{t,s}$ denote the dollar amount invested in shares of the representative firm,[17] an agent's financial wealth $W_{t,s}$ evolves according to

$$dW_{t,s} = \left(r_t W_{t,s} + \theta_{t,s}\left(\mu_t - r_t\right) + y_{t,s} + \pi W_{t,s} - c_{t,s}\right) dt + \theta_{t,s}\sigma_t dB_t, \quad W_{s,s} = 0. \tag{8}$$

1.4 Equilibrium

The definition of equilibrium is standard. An equilibrium is given by a set of adapted processes $\{c_{t,s}, \theta_{t,s}, r_t, \mu_t, \sigma_t\}$ such that *(i)* the processes $c_{t,s}$ and $\theta_{t,s}$ maximize an agent's objective (1) subject to the dynamic budget constraint (8), and *(ii)* markets for goods clear, i.e., $\int_{-\infty}^{t} \pi e^{-\pi(t-s)} c_{t,s} ds = Y_t$, and markets for stocks and bonds clear as well: $\int_{-\infty}^{t} \pi e^{-\pi(t-s)} \theta_{t,s} ds = S_t$, $\int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left(W_{t,s} - \theta_{t,s}\right) ds = 0$.

[17] We assume standard square integrability and transversality conditions. See, e.g., Karatzas and Shreve (1998) for details.

2 Homogeneous preferences: Overlapping generations and the risk-free rate

When agents have homogeneous preferences, it is possible to give an explicit characterization of the equilibrium interest rate and the stock-market price process.

Proposition 1 *Suppose that all agents have the same preferences, and consider the following non-linear equation for r:*

$$r = \rho + (1-\alpha)(\mu_Y + \pi(1-\beta)) - \gamma(2-\alpha)\frac{\sigma_Y^2}{2}, \tag{9}$$

where

$$\beta = \omega\overline{h}\left(\int_0^\infty G(u)e^{-(r+\pi+\gamma\sigma_Y^2-\mu_Y)u}du\right)\left(\pi + \frac{\rho}{1-\alpha} - \frac{\alpha}{1-\alpha}\left(r + \frac{\gamma}{2}\sigma_Y^2\right)\right). \tag{10}$$

Suppose that $\bar{r}$ is a root of (9) and $\bar{r} > \mu_Y - \gamma\sigma_Y^2$.[18] *Then there exists an equilibrium where the interest rate, the expected return, and the volatility of the stock market are all constant and given, respectively, by $r_t = \bar{r}$, $\mu_t = r + \gamma\sigma_Y^2$, and $\sigma_t = \sigma_Y$.*

In the equilibrium of Proposition 1 the equity premium and the volatility of the stock market in our OLG economy are identical to the respective quantities in a standard, infinitely-lived representative-agent model with the same preferences. (See, e.g., Weil (1989).) However, the interest rate is not. In the infinitely-lived representative-agent model the interest rate is given by $r = \rho + (1-\alpha)\mu_Y - \gamma(2-\alpha)\frac{\sigma_Y^2}{2}$, while equation (9) contains the additional term $(1-\alpha)\pi(1-\beta)$.

To see intuitively what drives this difference, it is helpful to ignore aggregate uncertainty for a moment ($\sigma_Y = 0$), so that $dY_t = \mu_Y Y_t dt$. Since aggregate consumption is given by $C_t = \pi\int_{-\infty}^t e^{-\pi(t-s)}c_{s,t}ds$, differentiating C_t with respect to t gives

$$\dot{C}_t = \pi c_{t,t} - \pi C_t + \pi\int_{-\infty}^t e^{-\pi(s-t)}\dot{c}_{s,t}ds, \tag{11}$$

[18] Lemma 1 contains sufficient — but not necessary — conditions on the parameters for the existence of such a root.

where we have used the short-hand notation $\dot{x} = \frac{dx}{dt}$. In the absence of uncertainty, the Euler equation of an agent who has access to annuities is given by[19] $\frac{\dot{c}_{s,t}}{c_{s,t}} = \frac{1}{1-\alpha}(r_t - \rho)$. Using this Euler equation inside (11) and re-arranging leads to

$$r_t = \rho + (1-\alpha)\left[\mu_Y + \pi\left(1 - \frac{c_{t,t}}{C_t}\right)\right]. \tag{12}$$

The interest rate in equation (12) is different from the respective interest rate in an identical economy but featuring an infinitely lived agent, which is given by $r = \rho + (1-\alpha)\mu_Y$. The source of the difference is that in an OLG economy only the Euler equation (and hence the per-capita consumption growth) of *existing* agents matters. The per-capita consumption growth of existing agents is in general different from the growth rate of *aggregate* consumption, because of deaths and births: In the absence of births, deaths would imply that per-capita consumption would be growing at the rate $\mu_Y + \pi$, simply because a fraction π of the population perishes per unit of time. However, births imply that a fraction of aggregate consumption accrues to arriving agents. Collectively, these agents consume $\pi c_{t,t}$, which is a fraction $\frac{\pi c_{t,t}}{C_t}$ of aggregate consumption. Therefore, the combined effect of births and deaths is that the per capita consumption growth of existing agents is $\mu_Y + \pi\left(1 - \frac{c_{t,t}}{C_t}\right)$.

As we show in the appendix, when all agents have homogeneous preferences, $\frac{c_{t,t}}{C_t}$ is constant and equal to β. Hence equation (12) simplifies to $r = \rho + (1-\alpha)[\mu_Y + \pi(1-\beta)]$, which coincides with Equation (9) when $\sigma_Y = 0$.

In the general case $\sigma_Y \neq 0$, the third term in Equation (9) accounts for the effects of precautionary savings, and it is the same in both the OLG and the infinitely-lived-agent economies.

In an influential paper, Weil (1989) pointed out that the standard representative-agent model cannot account for the low level of the risk-free rate observed in the data. Motivated by the low estimates of the IES in Hall (1988) and Campbell and Mankiw (1989), Weil's reasoning was that such values of the IES would lead to interest rates that are substantially higher than the ones observed in the data. Weil referred to this observation as the "low

[19] See, e.g., Blanchard (1985).

risk-free rate puzzle". In light of Weil's observation, we investigate whether the interest rate in our OLG economy is lower than in the respective infinitely-lived, representative-agent economy. The next lemma addresses this question.

Lemma 1 *Let* $\chi \equiv \rho + \pi - \alpha\left(\mu_Y - \frac{\gamma}{2}\sigma^2\right)$ *and assume that* $\chi > \pi$, $\mu_Y - \gamma\frac{\sigma_Y^2}{2} > 0$, *and*

$$\frac{1}{\omega} < \frac{\chi\int_0^\infty G(u)e^{-\chi u}du}{\pi\int_0^\infty G(u)e^{-\pi u}du}. \tag{13}$$

Then $\beta > 1$ *and hence the interest rate given by (9) is lower than the respective interest rate in an economy featuring an infinitely-lived representative agent.*

Whether condition (13) holds or not depends crucially on the life-cycle path of earnings. The easiest way to see this is to assume that[20] $\omega\chi > \pi$ and restrict attention to the parametric case $G(u) = e^{-\delta u}$, so that condition (13) simplifies to $\delta > \frac{\chi\pi(1-\omega)}{\omega\chi-\pi}$. Hence, the interest rate is lower in the overlapping generations economy as long as the life cycle path of earnings is sufficiently downward-sloping. The intuition for this finding is that agents who face a downward-sloping path of labor income need to save for the latter years of their lives. The resulting increased supply of savings lowers the interest rate. This insight is due to Blanchard (1985), who considered only the deterministic case and exponential specifications for $G(u)$. Condition (13) generalizes the results in Blanchard (1985). In particular, it allows $G(u)$ to have any shape, potentially even sections where the life-cycle path of earnings is increasing.

Next, we perform numerical exercises to gauge the quantitative effect of the OLG feature on the interest rate. For these exercises we use the life-cycle path of earnings $G(u)$ estimated by Hubbard et al. (1994), and set $\mu_Y = 0.02$ and $\sigma_Y = 0.041$, so that time-integrated data from our model can roughly reproduce the first two moments of annual consumption growth. We note that due to time integration, a choice of instantaneous volatility $\sigma_Y = 0.041$ corresponds to a volatility of 0.033 for model-implied, time-integrated, yearly consumption data, consistent with the long historical sample of Campbell and Cochrane (1999). The parameter π controls the birth-and-death rate, and we set $\pi = 0.02$ so as to match the birth

[20]This condition holds for the quantitative exercises we perform below.

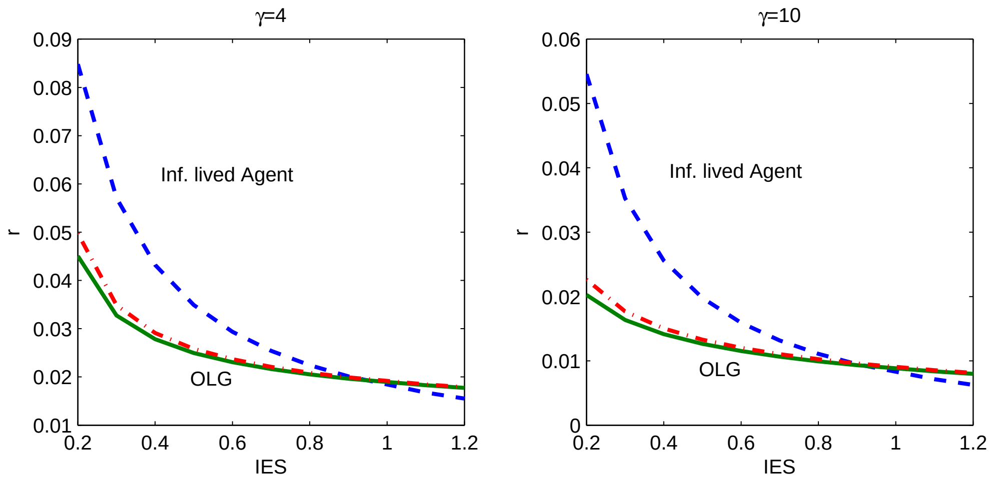

Figure 1: Interest rates when all agents have identical preferences. The dashed line pertains to an economy populated by an infinitely-lived agent, while the other two lines to our OLG model. The continuous line obtains when using the exact path for $G(u)$ reported in Hubbard et al. (1994) and performing the integration in Equation (10) numerically. The dash-dot line obtains when using non-linear least squares to project $G(\cdot)$ on a sum of scaled exponential functions, and then calculating the integral in Equation (10) exactly.

rate in the US population.[21] We choose a low, but positive, value of $\rho = 0.005$. In the calculations that follow, the choice $\rho = 0.005$ helps us illustrate that Weil's risk-free rate puzzle prevails even for low levels of the subjective discount rate if the economy is populated by a single representative agent. Finally, we choose $\omega = 0.88$ to match the fact that dividend payments and net interest payments to households are $1 - \omega = 0.12$ of personal income.[22]

Figure 1 reports the resulting interest rates for a) our OLG economy and b) the same economy featuring a single infinitely-lived representative agent. The figure shows that the

[21]Source: U.S. National Center for Health Statistics, Vital Statistics of the United States, and National Vital Statistics Reports (NVSR), annual data, 1950-2006. We add the net immigration rate (0.0024) to the birth rate in order to obtain the gross entry of "new agents" in the economy. We remark that in the data the birth rate and the death rate differ by about eighty basis points. For robustness, we also computed the model by matching the death rate rather than the birth rate, and obtained similar results.

[22]Source: Bureau of Economic Analysis, National income and product accounts, Table 2.1., annual data, 1929-2009. We combine dividend and net interest payments, in order to capture total flows from the corporate to the household sector. We note that the choice of $1 - \omega = 0.12$ is consistent with the gross profit share of GDP being about 0.3, since the share of output accruing to capital holders is given by the gross profit share *net of the investment share*. As a result, in our endowment economy that features no investment, it seems appropriate to match the parameter ω directly to the fraction of national income that accrues in the form of dividends and net interest payments.

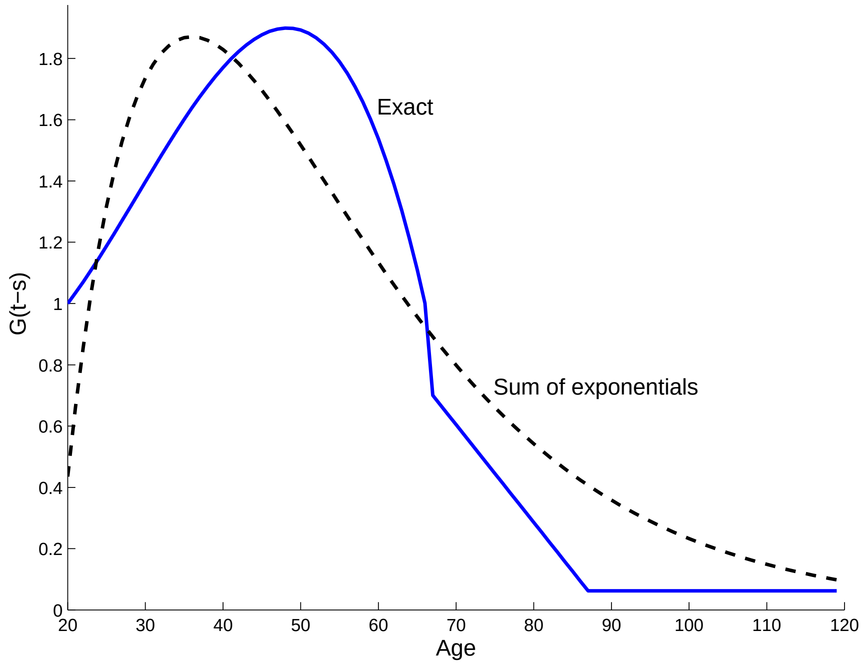

Figure 2: Hump-shaped profile of earnings over the life-cycle. The continuous line reports the profile estimated by Hubbard et al. (1994), while the broken line depicts the non-linear least-squares projection of the earnings profile in the data on a sum of scaled exponentials $G(u) = B_1 e^{-\delta_1 u} + B_2 e^{-\delta_2 u}$. The estimated coefficients are $B_1 = 30.72, B_2 = -30.29, \delta_1 = 0.0525$, and $\delta_2 = 0.0611$.

interest rate is generally lower in our economy.

We conclude this section with a technical remark: For the computation of the interest rate in Figure 1 (solid line), we used the exact path of life-cycle earnings reported in Hubbard et al. (1994) and evaluated the integral in equation (10) numerically. A convenient alternative is to approximate the hump-shaped path of life-cycle earnings by using a linear combination of exponential functions,

$$G(u) = B_1 e^{-\delta_1 u} + B_2 e^{-\delta_2 u}, \tag{14}$$

as in Figure 2, and then perform the integration in Equation (10) analytically. As Figure 2 shows, parameterizing $G(u)$ as in (14) preserves the humped shape of life-cycle earnings, while Figure 1 shows that there is practically no difference in terms of the resulting interest rate between the two approaches. However, the parametric alternative is substantially more tractable analytically in the presence of preference heterogeneity, which we introduce next.

3 Heterogeneous preferences: Expected utility, CRRA

The previous section shows that overlapping generations can generate low levels of the interest rate, even for low levels of the IES. However, with homogeneous preferences the equity premium and volatility of the stock market are the same as in the standard, infinitely-lived, representative-agent model, which faces well known problems in matching these quantities.

Motivated by these shortcomings, in the remainder of the paper we study preference heterogeneity. In this section we consider the special case in which all agents maximize CRRA expected utilities. However, agents can have different risk-aversion parameters. Starting our analysis of heterogeneity with the special case of CRRA utilities serves two goals. First, it allows us to relate to the existing literature and illustrate how our OLG structure leads to a stationary equilibrium. Second, it acts as a useful reference point for the next section, where we separate heterogeneity in risk aversion from heterogeneity in the IES.

The model is as follows. We continue to assume the functional-form specification (2) for the preferences of all agents. However, agents do not all have the same coefficient of risk aversion. In the interest of parsimony, we assume that at every point in time a proportion $\upsilon^A \in (0,1)$ of newly born agents have risk-aversion coefficient γ^A, while the remainder $\upsilon^B = 1 - \upsilon^A$ have risk-aversion coefficient γ^B. The parameters α^i, $i \in \{A, B\}$, satisfy the restriction $\alpha^i = 1 - \gamma^i$, so that the preferences given by (2) are equivalent to CRRA utilities. For the rest of the paper we maintain the convention $\gamma^A \leq \gamma^B$, and use superscripts to denote the type $i \in \{A, B\}$ of an agent.

A key quantity for our analysis is the fraction of aggregate output collectively consumed by type-A agents, defined as

$$X_t \equiv \frac{\upsilon^A \pi \int_{-\infty}^{t} e^{-\pi(t-s)} c_{t,s}^A ds}{Y_t}. \tag{15}$$

We also define the "market price of risk" (or Sharpe ratio) as

$$\kappa_t \equiv \frac{\mu_t - r_t}{\sigma_t}. \tag{16}$$

The next proposition presents the key results of this section.

Proposition 2 *Let X_t and $G(t-s)$ be defined as in Equations (15) and (14) respectively, and let $\beta_t^i \equiv \frac{c_{t,t}^i}{Y_t}$, $i \in \{A, B\}$, denote the consumption of a newly-born agent of type i as a fraction of aggregate consumption. Finally, let $\Gamma(X_t)$, $\omega^A(X_t)$, $\omega^B(X_t)$, and $\Delta(X_t)$ be defined as*

$$\Gamma(X_t) \equiv \left(\frac{X_t}{\gamma^A} + \frac{1-X_t}{\gamma^B}\right)^{-1}, \tag{17}$$

$$\omega^A(X_t) \equiv \frac{X_t}{\gamma^A}\Gamma(X_t), \;\; \omega^B(X_t) \equiv 1 - \omega^A(X_t), \tag{18}$$

$$\Delta(X_t) \equiv \omega(X_t)\left(\frac{\gamma^A+1}{\gamma^A}\right) + (1-\omega(X_t))\left(\frac{\gamma^B+1}{\gamma^B}\right), \tag{19}$$

and assume functions $g^i(X_t)$ and $\phi^j(X_t)$, for $i \in \{A, B\}, j \in \{1, 2\}$, that solve the system of ordinary differential equations (61) and (64) in the appendix. Then there exists an equilibrium in which X_t is a Markov diffusion with dynamics $dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dB_t$ given by

$$\sigma_X(X_t) = X_t\left(\frac{\Gamma(X_t)}{\gamma^A} - 1\right)\sigma_Y, \tag{20}$$

$$\mu_X(X_t) = X_t\left[\frac{r(X_t)-\rho}{\gamma^A} + \frac{\kappa^2(X_t)}{2}\frac{\gamma^A+1}{(\gamma^A)^2} - \pi - \mu_Y\right] + \upsilon^A\pi\beta^A(X_t) - \sigma_Y\sigma_X(X_t). \tag{21}$$

κ_t and r_t are both functions of X_t, given by

$$\kappa(X_t) = \Gamma(X_t)\,\sigma_Y, \tag{22}$$

$$r(X_t) = \rho + \Gamma(X_t)\left[\mu_Y - \pi\left(\sum_{i\in\{A,B\}} \upsilon^i \beta^i(X_t) - 1\right)\right] - \frac{\sigma_Y^2}{2}\left(\Gamma(X_t)\right)^2 \Delta(X_t), \tag{23}$$

and $\beta^i(X_t) = g^i(X_t)(\phi^1(X_t) + \phi^2(X_t))$ for $i \in \{A, B\}$.

In order to relate to existing work on heterogeneity, we focus our discussion of Proposition 2 on the market price of risk and the dynamics of X_t. We leave a discussion of the interest rate for Section 4.2.

As has been established in the literature,[23] the market price of risk in an economy with heterogeneous CRRA agents is identical to the market price of risk in an otherwise identical economy populated by a single, fictitious, expected-utility-maximizing "representative" agent with risk aversion given by $\Gamma(X_t)$. Indeed, equation (22) has the familiar form one encounters in single-agent setups:[24] the market price of risk is the product of the representative agent's risk aversion $\Gamma(X_t)$ multiplied by the volatility of consumption.

Equation (17) defines the risk aversion of the representative agent. It states that its inverse $\Gamma(X_t)^{-1}$, known as the "risk tolerance", is the consumption-weighted average of the risk tolerances of the two types of agents. An immediate implication is that the risk aversion $\Gamma(X_t)$ is a weighted average of individual risk aversions, and thus lies between γ^A and γ^B. The weights are given by $\omega^i(X_t)$:

$$\Gamma(X_t) \equiv \omega^A(X_t)\,\gamma^A + \omega^B(X_t)\,\gamma^B. \tag{24}$$

A further interesting implication of Equation (17) is that the representative agent does not have constant relative risk aversion, but instead her relative risk aversion is time-varying and countercyclical. Since $\Gamma(X_t) > \gamma^A$ for $X_t \in (0, 1)$, equation (20) implies that $\sigma_X > 0$. Hence, positive innovations to aggregate consumption increase the consumption share of

[23] See, e.g., Dumas (1989), Chan and Kogan (2002), and Zapatero and Xiouros (2010).
[24] See, e.g., Campbell and Cochrane (1999).

type-A agents. At the same time, $\Gamma(X_t)$ is a declining function of X_t, so that whenever aggregate consumption experiences a positive innovation, $\Gamma(X_t)$ declines.

The economic intuition behind this result is straightforward. Less risk-averse agents (type-A agents) invest more heavily in stocks than more risk-averse agents (type-B agents). As a result, a positive aggregate shock raises the wealth and consumption shares of less risk-averse agents. When less risk-averse agents own a larger fraction of aggregate wealth, the market price of risk is low, since these agents require a relatively smaller compensation for holding risk. Conversely, a negative shock increases the consumption and wealth shares of more risk averse agents, and consequently the market price of risk.

A side effect of risk-aversion heterogeneity in models with infinitely-lived agents is that eventually one type of agents "dominates" the economy. The reason is that the mean growth rate of wealth differs across agents, depending on their risk aversion. Hence, in the long run only one type of agents holds the entire wealth. That type of agents consumes the entire aggregate endowment and determines asset prices.[25]

A novel feature of our model is the existence of a non-trivial stationary distribution of X_t, owing to the OLG structure. To see this, note that, as $X_t \to 0$, Equations (20) and (21) imply that $\sigma_X(X_t) \to 0$, but $\mu_X(X_t) \to \upsilon^A \pi \beta^A(0) > 0$. Similarly, as $X_t \to 1$, $\sigma_X(X_t) \to 0$ and $\mu_X(X_t) \to -\upsilon^B \pi \beta^B(1) < 0$. As a result the process for X_t does not get absorbed at the points 0 or 1.[26] Intuitively, although the mean growth rates of wealth differ across agents, eventually all agents perish, regardless of their accumulated financial wealth. All newly-born agents enter the economy with no financial wealth. Their endowments at birth are equal and consist of earnings that are proportional to the level of output. As a result, each group of agents receives a minimum inflow of new members whose consumption is a non-zero fraction of aggregate output, ensuring that no type of agents dominates the economy.

[25] See, e.g., Dumas (1989) and Cvitanic and Malamud (2010).

[26] Technical details on boundary behavior and an analytic approach for the computation of the stationary density are given in Karlin and Taylor (1981).

4 Heterogeneous recursive preferences

We now turn to the study of heterogeneous recursive preferences. This allows us to separate the effects of heterogeneity in risk aversion and heterogeneity in the IES. To this end, we let α^A and α^B, the coefficients determining agents' IES, be arbitrary.

We structure this section as follows: we discuss first the equilibrium market price of risk, then the interest rate, and lastly the volatility, equity premium, and predictability of returns.

4.1 The market price of risk

The case of recursive preferences is more complex than that of additive utility. For this reason, we find it helpful to use as starting point a simple three-period model, and show afterwards that the results hold in the general case. We show, in particular, that the share X_t of consumption of one type of agent continues to be a sufficient, Markovian state variable, and derive conditions under which the market price of risk is higher than in the CRRA case with the same risk-aversion parameters.

4.1.1 A three-date model

We assume here that there are three periods[27] $t = 0, 1, 2$ and that aggregate consumption C_t follows a random walk given by $\frac{C_{t+1}}{C_t} = \varepsilon_{t+1}$, where $\log(\varepsilon_{t+1})$ is an i.i.d. random shock with standard deviation σ_ε. There are two types of agents (A and B) and markets are complete. All type-A agents are identical to each other, and similarly for all type-B agents. For brevity, we will henceforth refer to all type-A agents as "agent A", and similarly for agents of type B. Agents have standard recursive preferences given by

$$V_t^i = \left((1-\beta)\left(c_t^i\right)^{\alpha^i} + \beta\left[\mathrm{E}_t\left(V_{t+1}^i\right)^{1-\gamma^i}\right]^{\frac{\alpha^i}{1-\gamma^i}}\right)^{\frac{1}{\alpha^i}}. \tag{25}$$

By analogy with the definition of X_t in Section 3, we define x_t as the proportion of

[27] We need at least three periods in order to illustrate the interaction between consumption allocation across different states and its implications for the consumption allocation across different periods.

consumption accruing to agent A, namely $x_t \equiv c_t^A / C_t$.

With recursive preferences an agent's marginal rate of substitution MRS_{t+1}^i is[28]

$$MRS_{t+1}^i \equiv \beta \left[\frac{V_{t+1}^i}{\left(E_t \left(V_{t+1}^i \right)^{1-\gamma^i} \right)^{\frac{1}{1-\gamma^i}}} \right]^{1-\alpha^i-\gamma^i} \left(\frac{c_{t+1}^i}{c_t^i} \right)^{-(1-\alpha^i)}. \tag{26}$$

An empirical shortcoming of Equation (26) is that the value function V_{t+1}^i is not observable. Therefore it is common in the literature to use the homothetic properties of (25) in order to relate V_{t+1}^i to the consumer's *total* wealth. Specifically, letting $\widetilde{W}_t^i$ denote a consumer's total wealth (comprising financial wealth and the present value of her earnings discounted using MRS^i), the ratio of her consumption to her total wealth g_t^i satisfies [29]

$$g_t^i \equiv \frac{c_t^i}{\widetilde{W}_t^i} = (1-\beta) \left(\frac{V_t^i}{c_t^i} \right)^{-\alpha^i}. \tag{27}$$

Solving for V_t^i from (27) and using the resulting expression inside (26) gives

$$MRS_{t+1}^i = \beta \left[E_t \left(\frac{c_{t+1}^i}{c_t^i} \left(g_{t+1}^i \right)^{-\frac{1}{\alpha^i}} \right)^{1-\gamma^i} \right]^{\frac{\alpha^i}{1-\gamma^i}-1} \left(g_{t+1}^i \right)^{-\frac{1-\alpha^i-\gamma^i}{\alpha^i}} \left(\frac{c_{t+1}^i}{c_t^i} \right)^{-\gamma^i}. \tag{28}$$

In equilibrium $MRS_{t+1}^A = MRS_{t+1}^B$ for all t. Combining this fact with (28), using the

[28] The marginal rate of substitution is the ratio of marginal valuations of consumption at times $t+1$ and t. Specifically, $MRS_{t+1}^i = \left(\frac{\partial V_t^i}{\partial c_{t+1}^i} \right) / \left(\frac{\partial V_t^i}{\partial c_t^i} \right) = \left(\frac{\partial V_t^i}{\partial V_{t+1}^i} \right) \times \left(\frac{\partial V_{t+1}^i}{\partial c_{t+1}^i} \right) / \left(\frac{\partial V_t^i}{\partial c_t^i} \right)$. Evaluation of the derivatives in this last expression leads to (26).

[29] See, e.g., Hansen et al. (2007). A simple derivation of this fact starts with the observation that Euler's theorem for homogenous functions implies the recursion $V_t^i = \frac{\partial V_t^i}{\partial c_t^i} c_t^i + E_t \left(\frac{\partial V_t^i}{\partial V_{t+1}^i} V_{t+1}^i \right)$. Furthermore, the fact that the total value of wealth must equal the present value of consumption implies the recursion $\widetilde{W}_t^i = c_t^i + E_t \left(MRS_{t+1}^i \widetilde{W}_{t+1}^i \right)$. Combining these two recursions and using the fact that $MRS_{t+1}^i = \left(\frac{\partial V_t^i}{\partial V_{t+1}^i} \right) \times \left(\frac{\partial V_{t+1}^i}{\partial c_{t+1}^i} \right) / \left(\frac{\partial V_t^i}{\partial c_t^i} \right)$ implies that $V_t = \frac{\partial V_t^i}{\partial c_t^i} W_t$. Using (25) to compute $\frac{\partial V_t^i}{\partial c_t^i}$ and re-arranging leads to (27).

definition of x_t, and re-arranging yields

$$\frac{\left[E_t\left(\frac{x_{t+1}}{x_t}\varepsilon_{t+1}\left(g^A_{t+1}\right)^{-\frac{1}{\alpha^A}}\right)^{1-\gamma^A}\right]^{\frac{\alpha^A}{1-\gamma^A}-1}}{\left[E_t\left(\frac{1-x_{t+1}}{1-x_t}\varepsilon_{t+1}\left(g^B_{t+1}\right)^{-\frac{1}{\alpha^B}}\right)^{1-\gamma^B}\right]^{\frac{\alpha^B}{1-\gamma^B}-1}}\frac{\left(g^A_{t+1}\right)^{-\frac{1-\alpha^A-\gamma^A}{\alpha^A}}\left(\frac{x_{t+1}}{x_t}\right)^{-\gamma^A}}{\left(g^B_{t+1}\right)^{-\frac{1-\alpha^B-\gamma^B}{\alpha^B}}\left(\frac{1-x_{t+1}}{1-x_t}\right)^{-\gamma^B}} = \frac{\varepsilon_{t+1}^{-\gamma^B}}{\varepsilon_{t+1}^{-\gamma^A}}. \tag{29}$$

A first property of optimal consumption allocations is that x_t is Markov, and in the special case $\gamma^A = \gamma^B$ it is deterministic. To show these claims we proceed by backwards induction. In the last period, $W^i_2 = c^i_2$ and accordingly $g^i_2 = 1$. Evaluating (29) at $t = 1$ and using $g^i_2 = 1$ implies that a) x_2 depends only on x_1 and ε_2 and b) in the special case where $\gamma^A = \gamma^B$, the right-hand side of (29) equals 1, and hence x_2 does not depend on the realization of ε_2. Because x_2 depends at most on x_1 and ε_2, g^i_1 is a function of x_1,[30] so that we can write $g^i_1 = g^i(x_1)$. Proceeding backwards to $t = 0$, and using $g^i_1 = g^i(x_1)$ inside (29) implies that x_1 is a function of x_0 and ε_1, and accordingly $g^i_0 = g^i(x_0)$. Moreover, in the special case $\gamma^A = \gamma^B$, x_1 depends exclusively on x_0.

A second implication of an optimal consumption allocation is a simple approximate expression for the market price of risk, which we derive next. Using Equation (28) to evaluate the elasticity of MRS^i_t with respect to ε_t at $\varepsilon_t = 1$ gives

$$\frac{d\log\left(MRS^A_t\right)}{d\log(\varepsilon_t)} = -\frac{1-\alpha^A-\gamma^A}{\alpha^A}\frac{\left(g^A\right)'}{g^A}x'_t - \gamma^A\frac{x'_t}{x_t} - \gamma^A, \tag{30}$$

$$\frac{d\log\left(MRS^B_t\right)}{d\log(\varepsilon_t)} = -\frac{1-\alpha^B-\gamma^B}{\alpha^B}\frac{\left(g^B\right)'}{g^B}x'_t + \gamma^B\frac{x'_t}{1-x_t} - \gamma^B, \tag{31}$$

where $(g^i)' = \frac{dg^i}{dx_t}$ and $x'_t = \frac{\partial x_t}{\partial \varepsilon_t}$.

[30] Since $g^i_2 = 1$, so that $\frac{V^i_2}{c^i_2} = (1-\beta)^{\frac{1}{\alpha^i}}$, we obtain from (25) that

$$\frac{V^i_1}{c^i_1} = (1-\beta)^{\frac{1}{\alpha^i}}\left(1+\beta\left[\mathrm{E}_1\left(\frac{x_2}{x_1}\varepsilon_2\right)^{1-\gamma^i}\right]^{\frac{\alpha^i}{1-\gamma^i}}\right)^{\frac{1}{\alpha^i}},$$

which is a function of x_1 because x_2 is a function of x_1 and ε_2. Given (27), it follows that g^i_1 is a function of x_1. A similar argument shows that, generally, if g^i_{t+1} is a function of x_{t+1}, then g^i_t is a function of x_t.

The definition of $\omega^A(x_t)$ and $\omega^B(x_t)$ in Proposition 2 implies that $\omega^A(x_t) + \omega^B(x_t) = 1$. Therefore, $MRS_t^A = MRS_t^B$ implies

$$\frac{d\log(MRS_t^A)}{d\log(\varepsilon_t)} = \frac{d\log(MRS_t^B)}{d\log(\varepsilon_t)} = \sum_{i\in A,B} \omega^i(x_t) \frac{d\log(MRS_t^i)}{d\log(\varepsilon_t)}. \tag{32}$$

Using (30) and (31) and noting that $\omega^A(x_t)\gamma^A + \omega^B(x_t)\gamma^B = \Gamma(x_t)$ and $\omega^A(x_t)\frac{\gamma^A}{x_t} - \omega^B(x_t)\frac{\gamma^B}{1-x_t} = 0$, we compute

$$\sum_{i\in A,B} \omega^i(x_t) \frac{d\log(MRS_t^i)}{d\log(\varepsilon_t)} \quad = \quad -\Gamma(x_t) - \sum_{i\in A,B} \omega^i(x_t) \left(\frac{1-\gamma^i-\alpha^i}{\alpha^i}\right) \frac{(g^i)'}{g^i} x_t'. \tag{33}$$

Letting $R_{t+1} - r$ be the excess return on the stock market, a fundamental result in asset pricing asserts $E_t\left[(R_{t+1} - r)MRS_{t+1}^i\right] = 0$, which implies

$$E_t(R_{t+1} - r) = -cov\left(R_{t+1} - r, \frac{MRS_{t+1}^i}{E_t MRS_{t+1}^i}\right). \tag{34}$$

Approximating R_{t+1} and $\frac{MRS_{t+1}^i}{E_t MRS_{t+1}^i}$ to the first order as log-linear functions[31] of ε_{t+1} implies $\kappa_t = \frac{E_t(R_{t+1}-r)}{\sigma_t} \approx -\frac{d\log(MRS_t^i)}{d\log(\varepsilon_t)}\sigma_\varepsilon$. Using this approximation for κ_t together with (33) gives

$$\kappa \approx \Gamma(x_t)\sigma_\varepsilon + \sum_{i\in A,B} \omega^i(x_t) \left(\frac{1-\gamma^i-\alpha^i}{\alpha^i}\right) \frac{(g^i)'}{g^i} x_t' \sigma_\varepsilon. \tag{35}$$

When $1 - \gamma^i - \alpha^i = 0$, Equation (35) states that the market price of risk is equal to $\Gamma(x_1)\sigma_\varepsilon$. This is the expression we obtained for the market price of risk in the CRRA case (Equation [22]). In the general case ($1 - \gamma^i - \alpha^i \neq 0$), there is an additional term in Equation (35), namely $\sum_{i\in A,B} \omega^i(x_t) \left(\frac{1-\gamma^i-\alpha^i}{\alpha^i}\right) \frac{(g^i)'}{g^i} x_t' \sigma_\varepsilon$. Since many standard asset-pricing models tend to produce a low market price of risk, we are interested in determining the conditions under which this additional term is positive.

For the purposes of developing intuition, it suffices to study this issue for $t = 1$. We

[31] These log-linearizations are $R_{t+1} \approx d_1 + \frac{\sigma_t}{\sigma_\varepsilon} log(\varepsilon_{t+1})$ and $\frac{MRS_{t+1}^i}{E_t MRS_{t+1}^i} \approx d_2 + \frac{d\log(MRS_t^i)}{d\log(\varepsilon_t)} log(\varepsilon_{t+1})$ for appropriate constants d_1, d_2.

proceed in two steps. First we characterize allocations when $\gamma^A = \gamma^B = \gamma$. We then use the insights from this special case to provide an answer to our question.

Lemma 2 *Suppose that* $\gamma^A = \gamma^B = \gamma$ *and* $\left[E\left(\varepsilon_2\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}} > 1$. *Then,* $\alpha^A < \alpha^B$ *is equivalent to each of the following statements: (i)* $x_2 < x_1$*; (ii)* $\frac{d\left(\frac{x_2}{x_1}\right)}{dx_1} > 0$*; (iii)* $\frac{d\left(\frac{1-x_2}{1-x_1}\right)}{d(1-x_1)} < 0$.

The intuition for statement *(i)* is simple. When agent A has the lower IES ($\alpha^A < \alpha^B$), her equilibrium consumption path is flatter than that of agent B. Since the (certainty equivalent of) aggregate consumption growth is positive — that is, $\left[E\left(\varepsilon_2\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}} > 1$ — agent B needs to absorb most of the expected growth in aggregate consumption. Accordingly, the consumption of agent A as a share of aggregate consumption declines between periods 1 and 2 (i.e., $\frac{x_2}{x_1} < 1$).

The intuition behind statement *(ii)* resembles that for statement *(i)*. The assumption $\alpha^A < \alpha^B$ implies that agent A's preference for a flat consumption path is stronger than agent B's. When agent A accounts for a small part of aggregate consumption in period 1 (i.e., x_1 is small), then in an optimal allocation she has a relatively flat consumption path between periods one and two. Such an allocation does not impose a significant burden on agent B, whose anticipated consumption growth is more or less unaffected by agent A's consumption allocation. However, if agent A accounts for a large fraction of consumption in period 1, then agent B's consumption path would become very steep if agent A's consumption path were to remain relatively flat across the two periods. Therefore, in equilibrium agent A needs to absorb a larger fraction of anticipated aggregate growth. More generally, the larger agent A's consumption weight in period one, the larger the fraction of anticipated aggregate growth she needs to absorb. This implies that the ratio $\frac{x_2}{x_1}$ should be small when x_1 is small and large when x_1 is large, leading to statement (*ii*). (Statement *(iii)* obtains by symmetry.)

Building on Lemma 2, we can now derive the following sufficient conditions for the market price of risk being higher than in the CRRA case.

Lemma 3 *Suppose that* $\left[E\left(\varepsilon_2\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}} > 1$ *and* $\gamma^A < \gamma^B$. *Then, when* $\left|\gamma^B - \gamma^A\right|$ *and* $\left|\alpha^A - \alpha^B\right|$ *are not too large, the term* $\sum_{i \in A,B} \omega^i\left(x_1\right)\left(\frac{1-\gamma^i-\alpha^i}{\alpha^i}\right)\frac{\left(g^i\right)'}{g^i}x_1'\sigma_\varepsilon$ *(evaluated at* $\varepsilon_1 = 1$*) is non-negative if either* *(i)* $\gamma^i + \alpha^i - 1 \geq 0$ *for* $i \in \{A, B\}$ *and* $\alpha^A \leq \alpha^B$, *or*

(ii) $\gamma^i + \alpha^i - 1 \leq 0$ *for* $i \in \{A, B\}$ *and* $\alpha^A \geq \alpha^B$.

We present the proof of Lemma 3 in the appendix, and provide here only the intuition. Let us consider case *(i)*; case *(ii)* can be analyzed symmetrically. Since $\gamma^A < \gamma^B$, agent A assumes more of the aggregate consumption risk than agent B, so her consumption share at time 1 is higher if ε_1 is higher: $\frac{\partial x_1}{\partial \varepsilon_1} > 0$. The intuition behind Lemma 2 implies that the increase in x_1 will then require agent A to absorb a larger fraction of expected aggregate growth between periods 1 and 2 (case [*ii*] of Lemma 2). In summary, agent A's *expected* consumption growth between periods one and two increases in response to a positive period-one shock. (Symmetrically, agent B's *expected* consumption growth declines in response to the time-1 reduction of her consumption share). Hence, the optimal consumption allocation implies that the period-1 shock has *persistence* for individual agents' consumption. When agents have preferences for early resolution of uncertainty ($\gamma^i + \alpha^i - 1 > 0$), they are averse to intertemporally correlated risks, and hence require additional compensation for bearing such risks.[32]

4.1.2 General results

We show here that the expressions and intuitions we obtained in the context of the stylized, three-period model can be generalized to the full continuous-time, OLG model. The next proposition gives an explicit characterization of the dynamics of X_t, the interest rate, and the market price of risk in the full model.

Proposition 3 *Consider the same continuous-time OLG setup as in Proposition 2, except that* $1 - \gamma^i - \alpha^i \neq 0$. *Let* X_t, β_t^i, $\omega^i(X_t)$, *and* $\Gamma(X_t)$ *be defined as in Proposition 2, define*

$$\Theta(X_t) \equiv \frac{X_t}{1-\alpha^A} + \frac{1-X_t}{1-\alpha^B}, \tag{36}$$

[32] This property is at the core of the "long-run risks" literature, initiated by Bansal and Yaron (2004). The next section discusses the relation to Bansal and Yaron (2004) in more detail.

and assume functions $g^i(X_t)$ and $\phi^j(X_t)$, for $i \in \{A,B\}, j \in \{1,2\}$, that solve the system of ordinary differential equations (61) and (64) in the appendix. Then there exists an equilibrium in which X_t is a Markov diffusion with dynamics $dX_t = \mu_X(X_t)dt + \sigma_X(X_t)dB_t$ given by

$$\sigma_X(X_t) = \frac{X_t\left(\Gamma(X_t) - \gamma^A\right)}{\frac{\Gamma(X_t)}{\gamma^B} X_t(1-X_t)\left[\frac{1-\gamma^A-a^A}{\alpha^A}\frac{g^{A\prime}}{g^A} - \frac{1-\gamma^B-a^B}{\alpha^B}\frac{g^{B\prime}}{g^B}\right] + \gamma^A}\sigma_Y, \tag{37}$$

$$\mu_X(X_t) = X_t\left[\frac{r(X_t)-\rho}{1-\alpha^A} + n^A(X_t) - \pi - \mu_Y\right] + \upsilon^A\pi\beta^A(X_t) - \sigma_Y\sigma_X(X_t), \tag{38}$$

with

$$\kappa(X_t) = \Gamma(X_t)\sigma_Y + \sum_{i\in\{A,B\}}\omega^i(X_t)\left(\frac{1-\gamma^i-a^i}{\alpha^i}\right)\frac{g^{i\prime}}{g^i}\sigma_X(X_t), \tag{39}$$

$$r(X_t) = \rho + \frac{1}{\Theta(X_t)}\left\{\mu_Y - \pi\left(\sum_{i\in\{A,B\}}\upsilon^i\beta^i(X_t) - 1\right)\right\} \tag{40}$$

$$-\frac{1}{\Theta(X_t)}\left[X_t n^A(X_t) + (1-X_t)n^B(X_t)\right], \tag{41}$$

and $n^i(X_t)$ given by

$$n^i(X_t) = \frac{2-\alpha^i}{2\gamma^i(1-\alpha^i)}\kappa^2(X_t) + \frac{\alpha^i+\gamma^i-1}{2\gamma^i\alpha^i}\left(\frac{g^{i\prime}}{g^i}\sigma_X(X_t)\right)^2 - \frac{\gamma^i-\alpha^i(1-\gamma^i)}{\gamma^i(1-\gamma^i)}\frac{\alpha^i+\gamma^i-1}{(1-\alpha^i)\alpha^i}\left(\frac{g^{i\prime}}{g^i}\sigma_X(X_t)\right)\kappa(X_t), \tag{42}$$

and $\beta^i(X_t) = g^i(X_t)(\phi^1(X_t) + \phi^2(X_t))$ for $i \in \{A,B\}$.

The exact expression for the market price of risk $\kappa(X_t)$ (Equation [39]) in our continuous-time model coincides with the approximate expression we obtained in the three-period model (Equation [35]).

The following corollary to Proposition 3 confirms that also in the full model risk-aversion heterogeneity ($\gamma^A \neq \gamma^B$) is essential in order for X_t, and therefore the market price of risk and interest rates, to be stochastic. Heterogeneity in the IES does not suffice.

Corollary 1 *Consider the setup of Proposition 3, and impose the restriction $\gamma^A = \gamma^B = \gamma$,*

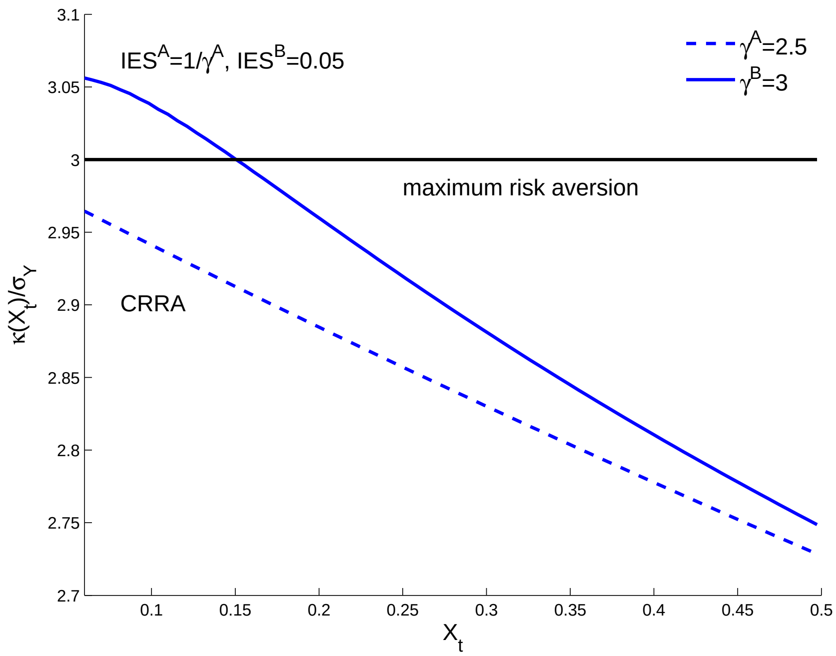

Figure 3: An illustration of Propositions 3 and 4. For low values of X_t, the ratio $\frac{\kappa(X_t)}{\sigma_Y}$ may be higher than the risk aversion of even the most risk averse agent.

(but $1-\alpha^A \neq 1-\alpha^B$.) Then the market price of risk is given by the constant $\kappa = \gamma\sigma$, and there exists a steady state featuring a constant interest rate and a constant consumption share of type-A agents given by some value $\bar{X} \in (0,1)$.

Finally, the next proposition shows that the results of Lemma 3 continue to apply in the context of the general model.

Proposition 4 *Consider the same setup as in Proposition 3, and let $\bar{X}$ denote the stationary mean of X_t. Then, provided that $\left|\gamma^B - \gamma^A\right|$ and $\left|\alpha^A - \alpha^B\right|$ are not too large, and subject to technical parameter restrictions given in the appendix, $\kappa(\bar{X}) > \Gamma(\bar{X})\sigma_Y$ if either*

(i) $\gamma^i + \alpha^i - 1 \geq 0$ for $i \in A, B$ and $\alpha^A \leq \alpha^B$, or

(ii) $\gamma^i + \alpha^i - 1 \leq 0$ for $i \in A, B$ and $\alpha^A \geq \alpha^B$.

An implication of Section 3 is that if agents have CRRA utilities, the ratio $\frac{\kappa(X_t)}{\sigma_Y}$ is a weighted average of the risk-aversion coefficients in the economy, and it corresponds to

Combinations	(i)	(ii)	(iii)	(iv)	(v)
			$\gamma^A = 4, \gamma^B = 10$		
IES^A	0.25	0.25	0.25	0.25	0.85
IES^B	0.1	0.25	0.85	0.05	0.05
	(i')	(ii')	(iii')	(iv')	(v')
			$\gamma^A = 2, \gamma^B = 10$		
IES^A	0.5	0.5	0.5	0.5	0.85
IES^B	0.1	0.5	0.85	0.05	0.05

Table 1: Parameter combinations for Figures 4-6.

the risk aversion of the "representative agent". However, with recursive preferences $\frac{\kappa(X_t)}{\sigma_Y}$ reflects more than just agents' risk aversions and their consumption weights. Interestingly, Propositions 3 and 4 even allow for the possibility that $\frac{\kappa(X_t)}{\sigma_Y}$ exceeds any agent's risk aversion for some values of x_t. Figure 3 provides a numerical example where this is indeed the case. In this example, a researcher using a standard, expected-utility-maximizing, representative-agent model to infer the risk aversion of the representative agent (see, e.g., Ait-Sahalia and Lo (2000)) would obtain an estimate exceeding the maximum risk aversion in the economy.

Even though Proposition 4 applies when preference heterogeneity is not "too large", the numerical exercises in Figure 4 illustrate that the conclusion of the proposition holds even for large heterogeneity in preferences. The parameters for these quantitative exercises and others below are chosen as follows. We use the same values for π, μ_Y, σ_Y, and the same parametric specification (14) for $G(u)$ as in Figure 1. We normalize $\upsilon^A = 0.1$ so that our type-A agents correspond to the top 10% of the population ranked by risk tolerance. This normalization facilitates a comparison of our results to data on the "90/10" ratio of consumption-inequality studies.[33]

In order to obtain a market price of risk in the range of $0.25 - 0.35$ and a "90/10" ratio in the range of $2 - 3$, we set $\gamma^A = 4$ and $\gamma^B = 10$. For comparison, we also consider the case where risk-aversion heterogeneity is larger ($\gamma^A = 2, \gamma^B = 10$).

[33] In our computations, type-A agents are typically at the top of the consumption distribution. Hence, when $\upsilon^A = 0.1$, the "90/10" ratio is approximately equal to $\frac{\bar{X}/0.1}{(1-\bar{X})/0.9}$. See Krueger and Perri (2006) for data on the 90/10 ratio.

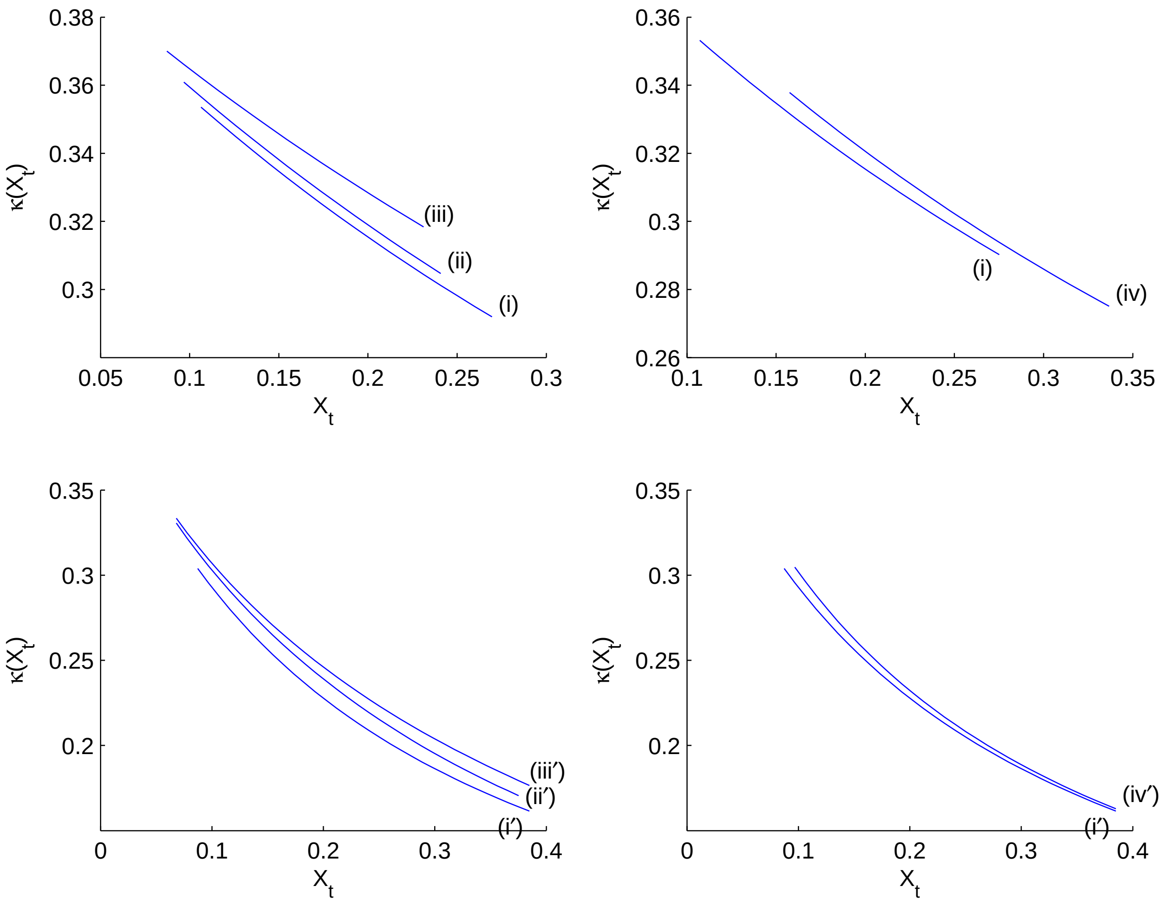

Figure 4: The market price of risk $\kappa(X_t)$ for the parameter combinations of the IES and the risk-aversion coefficients of agents A and B given in Table 1. For each combination of parameters, the consumption share of type-A agents X_t spans the interval between the bottom 0.5% and the top 99.5% percentiles of the stationary distribution of X_t.

To illustrate Proposition 4, we consider the IES-parameter combinations (i)-(iv) (respectively (i')-(iv') when $\gamma^A = 2, \gamma^B = 10$) in Table 1. Figure 4 depicts the resulting market price of risk for each parameter combination. We leave a discussion of combinations (v) and (v') for section 4.3.

For simplicity, in all combinations (i)-(iv) agent A's IES is set equal to the inverse of her risk-aversion coefficient, so that she has CRRA preferences. In combination (i) agent B also has expected utility preferences. For comparison, in combination (ii) agents differ only in their risk aversion, but not in their IES. Combination (iii) satisfies the requirements of case *(i)* of Proposition 4, and combination (iv) satisfies the requirements of case *(ii)* of Proposition 4. Consistent with Proposition 4, Figure 4 shows that line (iii) lies above line (i). Similarly, line (iv) lies above line (i). The bottom panel of Figure 4 shows that these conclusions continue to hold when $\gamma^A = 2, \gamma^B = 10$.

We conclude with a few remarks relating the results of this section to existing literature. In our model an increased market price of risk (compared to the case where agents have CRRA utilities) is driven by the non-i.i.d. nature of *individual* agents' consumption growth combined with appropriate preferences for the timing of the resolution of uncertainty. These features are reminiscent of the "long-run risks" literature initiated by Bansal and Yaron (2004). However, in our setup predictable consumption components arise endogenously and *only* at the level of individual agents' consumption.[34] In contrast to Bansal and Yaron (2004), aggregate consumption growth is i.i.d.. Also, in our setup there is no requirement for the IES of any agent to be above one. Finally, unlike Bansal and Yaron (2004), our framework does not require stochastic volatility of aggregate consumption in order to produce countercyclical variation in the market price of risk.

4.2 The interest rate

The expression for the interest rate when agents have either CRRA utilities (Equation [23]) or recursive preferences (equation [41]) has the same familiar structure as in the homogeneous preference case (Equation [9]). The first term in either of these expressions is the discount rate, the second term reflects the agents' attitudes toward intertermporal substitution and the third term captures precautionary savings motives. Comparing the second term of Equations (23) and (41), we note that the expression inside curly brackets (i.e., the consumption growth of existing agents) is no longer multiplied by a weighted average of agents' risk-aversion coefficients $\Gamma(X_t)$, but rather by $\frac{1}{\Theta(X_t)}$. By its definition, $\Theta(X_t)$ is simply a consumption-weighted average of agents' IES. This is a natural outcome. Because the recursive-preference specification disentangles agents' IES from their risk aversion, it

[34] See Malloy et al. (2009) for empirical evidence of such predictability.

helps clarify that *intertemporal* consumption-smoothing motives are controlled exclusively by agents' attitudes towards intertemporal substitution.

An important practical implication of Equation (41) is that the model allows for time variation in the interest rate that is unrelated to time variation in the anticipated growth rate of aggregate consumption. Specifically, aggregate consumption growth in our model is unpredictable, since $\mathrm{E}_t\left[\log\left(C_{t+\Delta}\right)\right] - \log(C_t) - \left(\mu_Y - \sigma_Y^2/2\right)\Delta = 0$ for any $\Delta > 0$. However, the interest rate is a function of X_t, which is in general time varying. Now suppose that an econometrician ignored heterogeneity and erroneously postulated that the economy is populated by a single agent who consumes the aggregate endowment. Then, as Vissing-Jorgensen (2002) shows, the discrete-time Euler equation of such an agent would imply the (approximate) relationship

$$IES = \frac{d\mathrm{E}_t\left[\log\left(C_{t+1}/C_t\right)\right]}{d\mathrm{E}_t\left[\log\left(1+r_{t+1}\right)\right]}, \tag{43}$$

where C_{t+1}/C_t is the aggregate consumption growth. Equation (43) forms the basis of empirical estimation exercises. Our model provides a simple case in which heterogeneity implies non-trivial time variation in the interest rate ($d\mathrm{E}_t\left[\log\left(1+r_{t+1}\right)\right] \neq 0$), while the aggregate expected consumption growth has none ($d\mathrm{E}_t\left[\log\left(C_{t+1}/C_t\right)\right] = 0$). Accordingly, the econometrician would erroneously conclude that the IES of the (assumed homogeneous) agents in the economy is zero.

The above discussion is consistent with the common finding that estimates of the IES obtained from aggregate consumption data (combined with single-representative agent assumptions) are indistinguishable from zero. This common finding contrasts with the higher estimates obtained from microeconomic data.[35] The literature routinely views these differences in the estimates as symptoms of limited participation by a fraction of the population in any form of asset market (e.g., bonds, savings accounts, stocks, mortgages etc.).[36] Our

[35] Compare the results in, e.g., Hall (1988) and Campbell and Mankiw (1989) with the results in, e.g., Vissing-Jorgensen (2002).

[36] The usual argument is that even though aggregate consumption includes the consumption of such "hand-to-mouth" agents, their consumption choices are not governed by any Euler equation.

analysis supports an even stronger result: Even if all agents have recursive preferences and participate in asset markets, it is erroneous to postulate the existence of a representative agent with constant coefficients of risk aversion and constant IES. Heterogeneity leads to time variation in the interest rate that is unrelated to time variation in predictable aggregate consumption growth, something that is impossible when the representative agent has standard recursive preferences as given in equation (25) (see, e.g., Campbell and Beeler (2009)). As a result, Equation (43) erroneously leads to a zero estimate of the IES when aggregate consumption growth is i.i.d., and, more generally, biases the estimated IES towards zero when aggregate consumption growth has time-varying predictable components.

4.3 The volatility and the equity premium

In this section we present a number of results concerning the volatility of returns and the equity premium. First, we show formally that the stock-market volatility may be higher than the dividend volatility only in the presence of risk-aversion heterogeneity; IES heterogeneity alone does not generate excess volatility. Second, for a range of plausible parameters, we document a trade-off between a high market price of risk and a high return volatility.

The following proposition addresses the possibility of excess volatility.

Proposition 5 *Consider the setup of Proposition 3, and impose the restriction* $\gamma^A = \gamma^B = \gamma$ *(but* $1 - \alpha^A \neq 1 - \alpha^B$*). Then* $\sigma_t = \sigma_Y$*.*

Proposition 5 is an immediate consequence of Corollary 1. In the absence of risk-aversion heterogeneity, the market price of risk is constant and the interest rate is deterministic. As a result there are no unexpected shocks to discount rates and the volatility of stock returns reflects exclusively the volatility of cash-flows.

In light of Proposition 5, we assume that $\gamma^A \neq \gamma^B$ for the rest of this section. Next, we study how the interaction between IES heterogeneity and risk-aversion heterogeneity affects the joint behavior of the market price of risk and return volatility. We start with a graphical illustration. Figure 5 graphs these quantities for the same cases as in Figure 4. We also include the parameter combinations (v) and (v') of Table 1, in order to illustrate a situation where agent A's IES (0.85) is larger than the reciprocal of her risk aversion, while agent B's IES (0.05) is smaller than the reciprocal of her (own) risk aversion.

In order to compare the volatility of our all-equity financed firm to the levered-equity returns observed in the real world, we follow the straightforward approach advocated by Barro (2006). Barro (2006) proposes to treat levered equity as a zero-net supply "derivative" security of the positive-supply unlevered equity. The introduction of a zero net supply claim leaves all allocations and prices unchanged, while the Modigliani-Miller theorem implies that levered equity has the same return as a (constantly rebalanced) replicating portfolio that is long unlevered equity and short debt. Letting $\frac{B_t}{S_t - B_t}$ denote the ratio of debt B_t over the value of levered equity $S_t - B_t$, the Modigliani-Miller theorem asserts that the excess rate of return of levered equity is $\left(1 + \frac{B_t}{S_t - B_t}\right)\left(\frac{dS_t}{S_t} - r_t dt\right)$. Following Barro (2006) we set $\frac{B_t}{S_t - B_t} = 0.5$ to reflect the average historical leverage ratio in NIPA data and report results for levered equity.

A first observation about the top-left subplot of Figure 5 is that the volatility is lowest in cases (ii) and (iii), that is, when the less risk-averse agent A has an IES equal to or smaller than that of agent B. The volatility increases as we move to case (i), where both agents have time-additive, CRRA preferences, so that the agent with the lower risk aversion (A) also has the higher IES. Cases (iv) and (v) illustrate that as agent A's IES remains equal to or becomes larger than the level implied by CRRA preferences, while agent B's becomes smaller than the level implied by CRRA preferences, the stock-market volatility increases further. Comparing cases (i)-(iv) in Figure 5 to the respective cases in Figure 4 shows that the parameter combinations that yield *higher* levels of the market price of risk are associated with *lower* values of volatility and vice versa. The top-right panel of Figure 5 shows that the same conclusions hold when risk-aversion heterogeneity is given by $\gamma^A = 2$ and $\gamma^B = 10$.

To help explain the patterns in Figure 5, Figure 6 analyzes why cases (ii) and (iii) exhibit lower levels of volatility than cases (iv) and (v).[37] The figure shows that even though in all cases the market price of risk exhibits similar counter-cyclical behavior (the dashed lines in

[37] The patterns are unchanged when $\gamma^A = 2$ and we omit the analog of Figure 6 for cases (i')-(v') to save space.

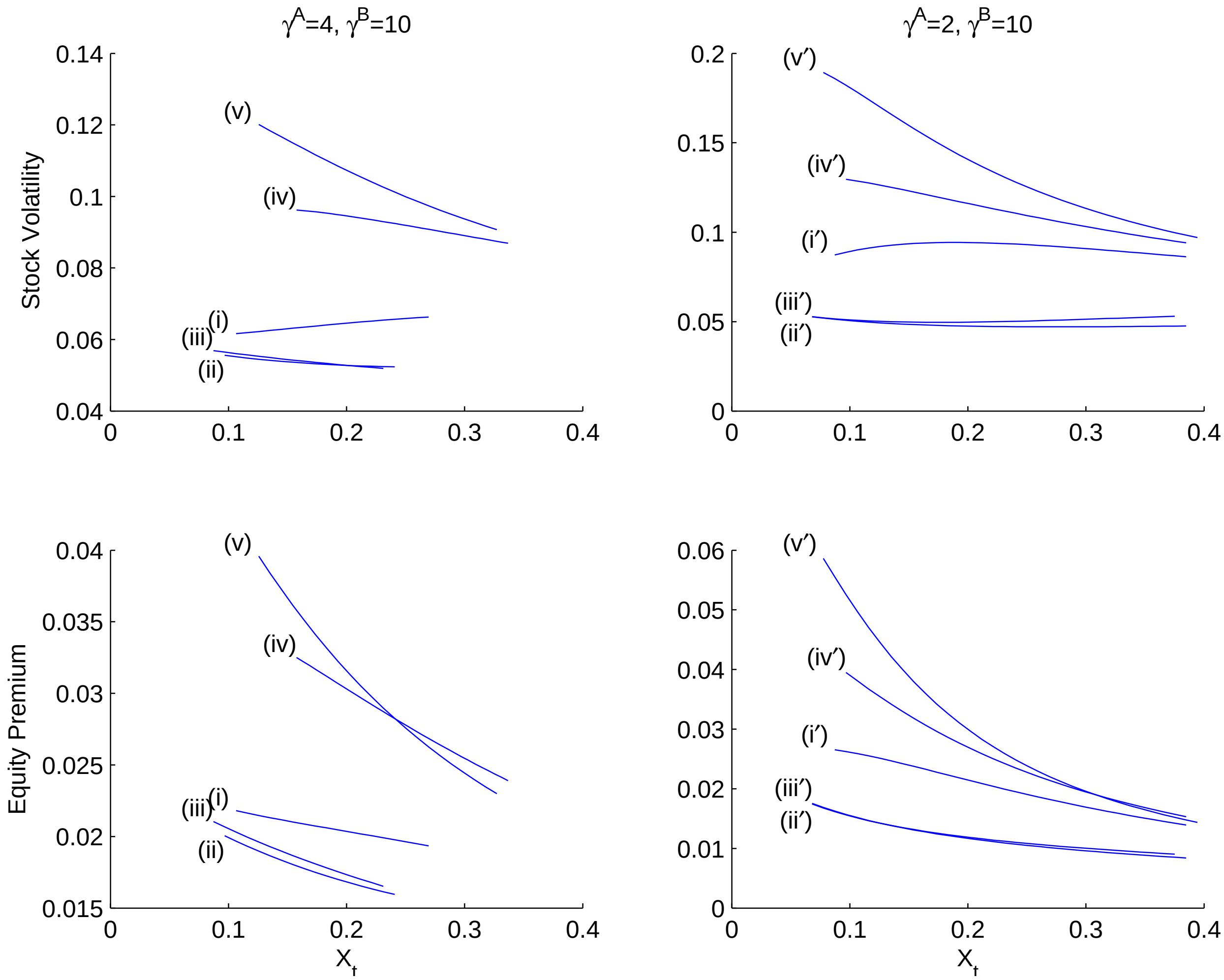

Figure 5: The stock market volatility (σ_t) and the equity premium ($\mu_t - r$) for the parametric combinations (i)-(v) and (i')-(v') given in Table 1. For each combination of parameters the range of values of the consumption share of type-A agents (X_t) spans the interval between the bottom 0.5% and the top 99.5% percentiles of its stationary distribution.

the figure are declining), the interest rate in cases (ii) and (iii) is procyclical (i.e., increasing in X_t), while in cases (iv) and (v) it is countercyclical (i.e., decreasing in X_t). As a consequence, in cases (ii) and (iii) a positive shock to the endowment increases X_t and lowers the market price of risk, but it also raises the interest rate, attenuating the overall countercyclicality of the discount rate. The volatility of returns is accordingly lower. By contrast, in cases (iv) and (v) the countercyclical interest rate reinforces the countercyclical market price of risk, resulting in a higher volatility of returns.

The behavior of the interest rate depicted in Figure 6 is intuitive. If agents A and B have the same IES (case (ii)), then, as the consumption weight (X_t) of the less risk-averse agents increases, precautionary savings in the economy fall. To restore bond-market clearing, the interest rate must rise, and hence the interest rate is procyclical. In case (iii) the lower IES of agent A *amplifies* the procyclical behavior of the interest rate: a higher X_t means an increased relative importance of type-A agents, who are simultaneously less inclined to save for precautionary reasons and more averse to intertemporal substitution (and hence require a higher interest rate to substitute consumption intertemporally). By contrast, in cases (i), (iv), and (v), the fact that agent A has a higher IES tends to *counteract* the reduction in precautionary savings as X_t increases. As a result, the interest rate becomes hump-shaped in case (i) and counter-cyclical in cases (iv) and (v).

It is useful to relate these observations to Gomes and Michaelides (2008), who also studies the interaction of IES heterogeneity and risk-aversion heterogeneity in a model in which capital supply is perfectly elastic, the price of capital (Tobin's q) is one, and the volatility of returns is primarily driven by exogenous depreciation shocks to the quantity (as opposed to the price) of capital. Although the many differences in the model details prevent an exact comparison, the findings of Gomes and Michaelides (2008) on the market price of risk are broadly consistent with case *(i)* of Proposition 4. Specifically, Gomes and Michaelides (2008) documents that their model generates an increased market price of risk when agents with low risk aversion also have low IES. As volatility is exogenous in their model, this increased market price of risk translates directly into a higher equity premium.

By contrast, we study an endowment economy, so that the price of capital is time varying and the volatility of the stock market is endogenous.[38] As already explained, our analytical results on the market price of risk support the conclusions of Gomes and Michaelides (2008). However, we find that once volatility is endogenized the preference specifications associated

[38] Admittedly, this comes at the cost of not being able to analyze capital-accumulation decisions. The usual justification for the simplifications allowed by endowment-based models is that capital accumulation decisions in the data exhibit less time variation than the price of capital. An interesting extension of the model would introduce capital accumulation subject to adjustment costs in order to model variation in both the price and the quantity of capital. However, such an extension is outside the scope of the current paper.

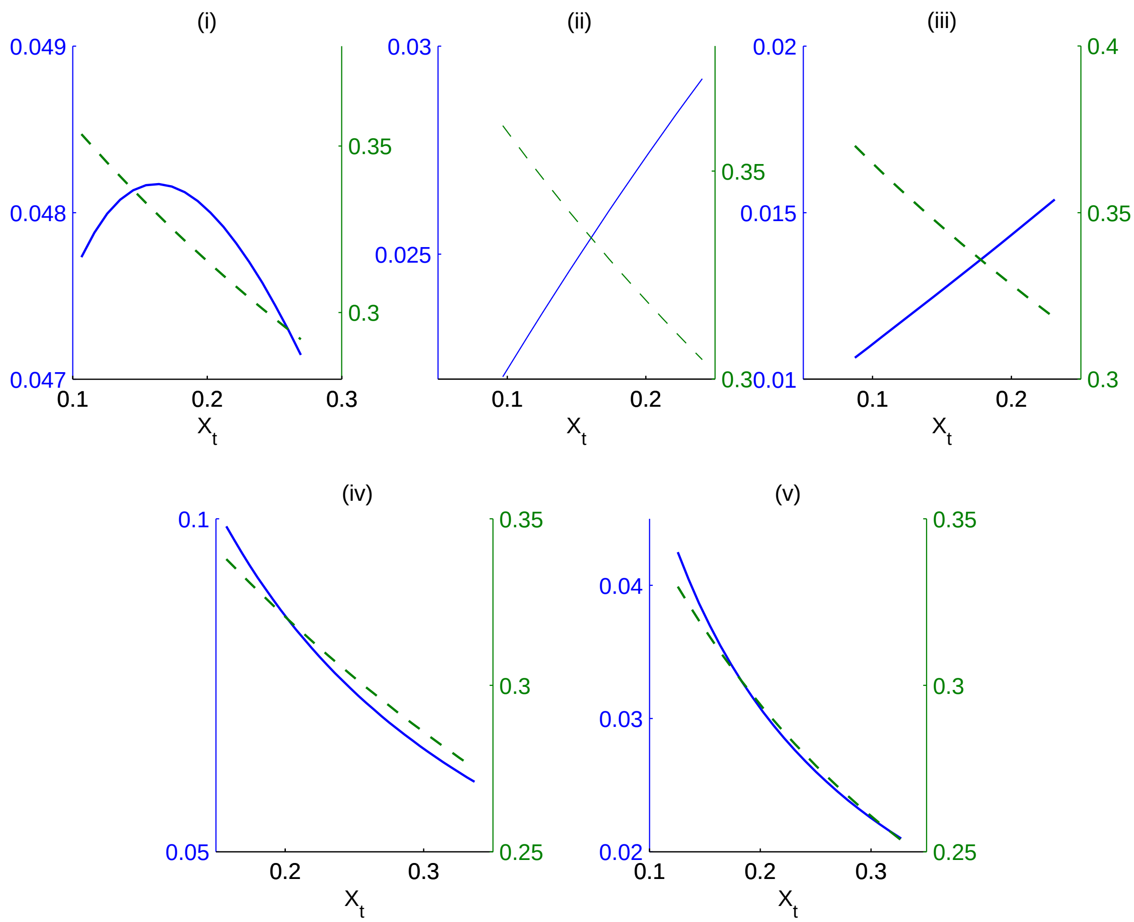

Figure 6: The interest rate r_t (solid line, depicted on the left y-axis) and the market price of risk κ_t (dashed line, depicted on the right y-axis) for the parametric combinations (i)-(v) given in table 1. For each combination of parameters, the range of values of the consumption share of type$-A$ agents X_t spans the interval between the bottom 0.5% and the top 99.5% percentiles of the stationary distribution of X_t.

with a higher market price of risk may also generate a lower volatility of returns, leaving the overall effect on the equity premium ambiguous.

Figure 7 gives an alternative view of the trade-offs in our model. Fixing the risk-aversion coefficients ($\gamma^A = 4, \gamma^B = 10$), the figure graphs six stationary moments for a range of choices for IES^A and IES^B, corresponding to the range of estimates obtained in micro studies.[39] The six stationary moments that we report are a) the equity premium, b) the volatility of

[39] See, e.g., Vissing-Jorgensen (2002).

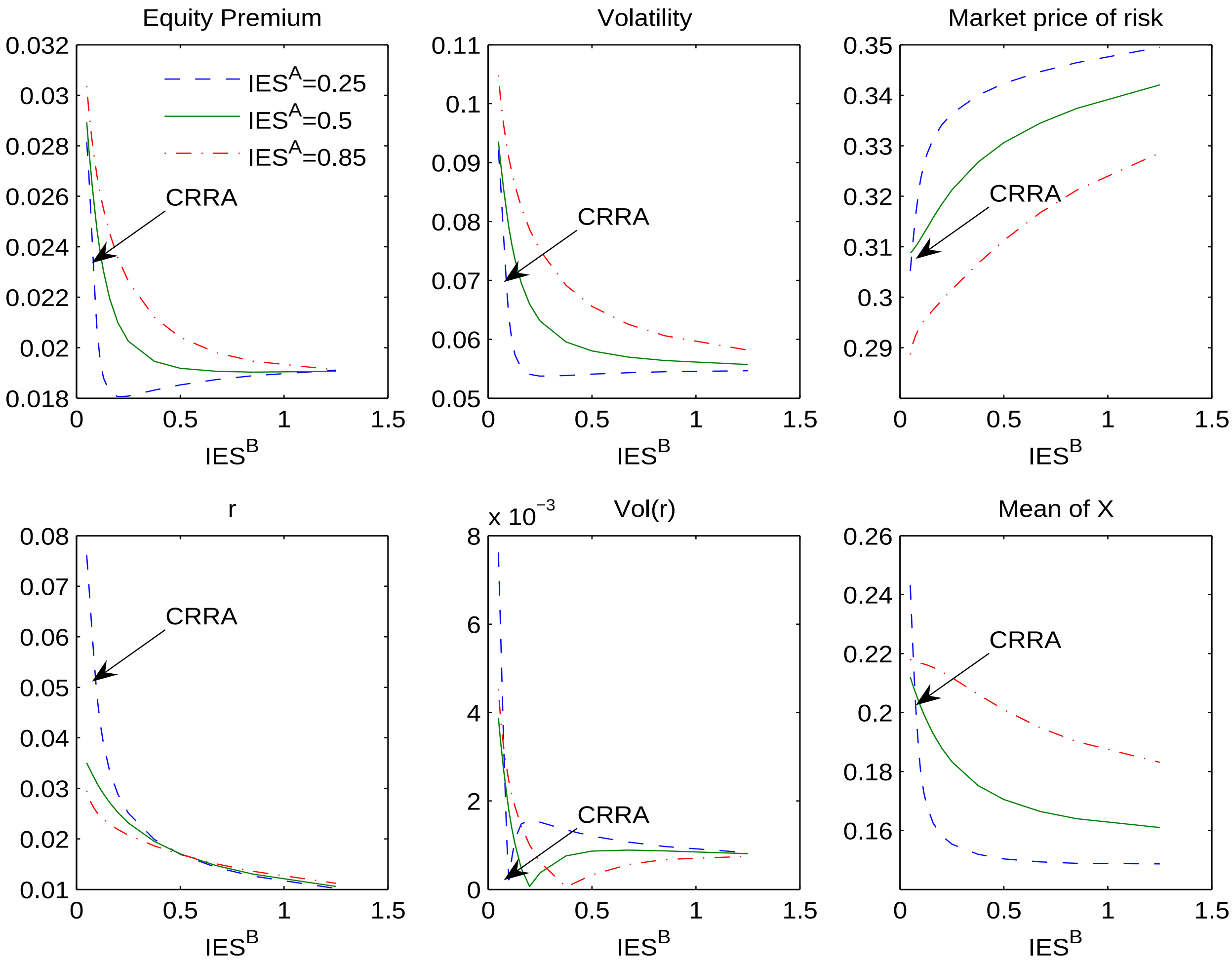

Figure 7: Stationary means of the equity premium, volatility, market price of risk, interest rate, volatility of interest rate, and consumption fraction accruing to type-A agents. The risk-aversion coefficient of agent A is 4 and of agent B is 10. The IES of agent B is on the x-axis. The three different lines in each plot correspond to IES^A values of 0.25, 0.5, and 0.85 respectively. The arrow points to the CRRA case ($IES^A = 0.25$ and $IES^B = 0.1$).

stock-market returns, c) the market price of risk, d) the interest rate, e) the volatility of the interest rate, and f) the stationary value of X_t.

There are several observations about Figure 7 worth highlighting. First, we note that, keeping IES^A fixed, the stock-market volatility declines with IES^B. This is simply an alternative illustration of the intuition we highlighted earlier: If the agents with the high risk aversion also have high IES, the interest rate tends to become procyclical. The procyclicality of the interest rate attenuates the countercyclicality of the market price of risk and the total discount rate, thus lowering return volatility. Second, we note that in general the market price of risk exhibits the opposite patterns from volatility. Contrary to the top-left subplot, the lines in the top-right subplot are upward sloping, and, symmetrically, for a given level of IES^B the market price of risk is declining in IES^A. We note that these opposing patterns are consistent with Proposition 4: Case *(i)* of that proposition states that, when $\gamma^i > 1-\alpha^i$ (as for almost all parametric combinations in Figure 7) and $\alpha^B > \alpha^A$ (equivalently, $IES^B > IES^A$), the market price of risk is higher than in the CRRA case. As we argued above, however, $IES^B > IES^A$ also implies a procyclical interest rate and lower volatility.

These opposing patterns of the market price of risk and volatility mitigate the model's ability to produce a high equity premium, which is the product of the two. The quantitative examples in Figure 7 show that volatility is the relatively more important component in this product, since the patterns of the equity premium mirror those of volatility. Figure 7 also suggests that the model's overall performance is best when agents with low risk aversion also have a high IES and vice versa. For instance, when $IES^A = 0.85$ and $IES^B = 0.05$ (case (v) of Table 1) the model generates an equity premium slightly above 3%, a volatility of 11%, a market price of risk of 0.29, a risk-free rate of 3%, and a standard deviation of the real interest rate below 0.5%. For completeness, Table 2 also reports the results of standard predictability regressions.

In conclusion, Figure 7 shows that, fixing unequal risk-aversion coefficients, heterogeneity in the IES can have important effects. In the numerical examples that we consider, IES heterogeneity can lead to improvements on the heterogeneous-CRRA specification, particularly when *(i)* agents with a lower risk aversion have a higher IES (as in the CRRA case), and *(ii)* the degree of IES heterogeneity is high — in particular, higher than in the CRRA case. Whether such a negative correlation between the risk-aversion and the IES is a feature of the joint distribution of these quantities in the data is a non-trivial empirical question, which we do not attempt to answer here.[40] Overall, the numerous parameter combinations depicted

[40]We simply note the suggestive evidence from micro data that agents with higher levels of wealth tend to exhibit higher levels of the IES. Inside our model this can only happen if the agents with the lower risk

Horizon	Data (Long Sample)		Model (i)		Model (v)	
(Years)	Coefficient	R^2	Coefficient	R^2	Coefficient	R^2
1	-0.13	0.04	-1.01	0.01	-0.22	0.01
			[-4.02, 0.31]	[0.00, 0.07]	[-1.04, 0.11]	[0.00, 0.06]
2	-0.28	0.08	-1.97	0.03	-0.43	0.02
			[-8.01, 0.58]	[0.00, 0.14]	[-2.12, 0.21]	[0.00, 0.12]
3	-0.35	0.09	-3.01	0.04	-0.65	0.03
			[-11.47, 1.03]	[0.00, 0.20]	[-3.04, 0.34]	[0.00, 0.18]
5	-0.60	0.18	-5.05	0.07	-1.06	0.05
			[-17.41, 1.80]	[0.00, 0.30]	[-4.56, 0.64]	[0.00, 0.28]
7	-0.75	0.23	-6.95	0.10	-1.47	0.07
			[-22.41, 2.68]	[0.00, 0.39]	[-6.06, 0.86]	[0.00, 0.37]

Table 2: Long-horizon regressions of excess returns on the log P/D ratio. To account for the well documented finite sample biases driven by the high autocorrelation of the P/D ratio, the simulated data are based on 1000 independent simulations of 106-year long samples, where the initial condition for X_0 for each of these simulation paths is drawn from the stationary distribution of X_t. For each of these 106-year long simulated samples, we run predictive regressions of the form $R_{t\to t+h} = \alpha + \beta \log (P_t/D_t)$, where h is the horizon for returns in years. We report the median values for the coefficient β and the R^2 of these regressions, as well as the $[2.5\%, 97.5\%]$ intervals.

in Figure 7 suggest that preference heterogeneity can help improve the performance of the textbook asset pricing model; however, it is unlikely that preference heterogeneity *alone* is the sole explanation of all asset-pricing puzzles.[41]

5 Concluding remarks

We analyzed the asset-pricing implications of preference heterogeneity in a framework combining overlapping generations and recursive utilities. We expressed the equilibrium in terms of the solution to a system of ordinary differential equations, and characterized properties of the solution.

aversion also have the higher IES, since agents with lower risk aversion are typically the richer ones. Indeed, as the bottom right subplot of Figure 7 shows, for any combination of IES^A and IES^B, less risk-averse agents are wealthier, since their fraction of the consumption distribution is higher than their mass in the population (0.1).

[41] In unreported results, we also ran simulations for cases with larger risk-aversion heterogeneity ($\gamma^A = 2, \gamma^B = 10$), as well as with different levels for IES^A. Although the volatility and the market price of risk tend to be substantively affected by these alternative specifications, we were not able to obtain an equity premium significantly higher than the levels in Figure7.

In particular, we showed that in the case where agents have heterogeneous CRRA preferences, the ratio of the market price of risk to the consumption-growth volatility is a countercyclical, stationary, weighted average of agents' risk aversions with weights that reflect their consumption shares. When agents have recursive preferences, however, this ratio is also affected by the endogenous persistence in individual agents' consumption growth caused by the optimal consumption sharing rule. We analyzed parameter conditions that lead to a higher market price of risk. In some cases, the resulting market price of risk may even be higher than in a world populated exclusively by the agent with the highest risk-aversion coefficient. We also showed that when agents have heterogeneous preferences, the common empirical approach of postulating a constant IES for the representative agent and using standard methods to estimate that IES typically leads to estimates biased towards zero. Finally, we separated the effects of risk-aversion heterogeneity and IES heterogeneity and found that risk-aversion heterogeneity is indispensable for the model to have interesting asset-pricing implications. However, we also found that the two dimensions of heterogeneity interact. Specifically, fixing some heterogeneous risk-aversion coefficients, the type and magnitude of IES heterogeneity can significantly affect the asset-pricing implications of the model.

One interesting extension of the present model that would improve its performance, while retaining the basic structure, would be to further exploit the OLG framework and allow for limited intergenerational risk sharing as in Gârleanu et al. (2009). We leave such an extension for future work.

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A Proofs

We start with the proof of Proposition 3. We then provide the proofs of Propositions 1 and 2 and Corollary 1 as special cases.

Proof of Proposition 3. We start by defining the constants

$$\Xi_1^i = -\frac{\alpha^i + \gamma^i - 1}{\alpha^i}, \quad \Xi_2^i = -\frac{\alpha^i}{(1-\alpha^i)(1-\gamma^i)}, \quad \Xi_3^i = -\frac{\rho+\pi}{\alpha^i}(1-\gamma^i), \quad \Xi_4^i = -\frac{\alpha^i + \gamma^i - 1}{(1-\alpha^i)(1-\gamma^i)},$$

where $i \in A, B$. Furthermore, we let

$$\frac{d\xi_t}{\xi_t} \equiv -r_t dt - \kappa_t dB_t \tag{44}$$

$$\frac{\tilde{\xi}_u}{\tilde{\xi}_t} \equiv e^{-\pi(u-t)} \frac{\xi_u}{\xi_t}. \tag{45}$$

Because existing agents have access to a frictionless annuity market and can trade dynamically in stocks and bonds without constraints, the results in Duffie and Epstein (1992) and Schroder and Skiadas (1999) imply that the optimal consumption process for agents with recursive preferences of the form (2) is given by

$$\frac{c_{u,s}^i}{c_{t,s}^i} = e^{\frac{1}{1-\alpha^i}\int_t^u f_V^i(w)dw} \left(\frac{(1-\gamma)V_{u,s}^i}{(1-\gamma)V_{t,s}^i}\right)^{\Xi_4^i} \left(\frac{\tilde{\xi}_u}{\tilde{\xi}_t}\right)^{\frac{1}{\alpha^i - 1}}. \tag{46}$$

From this point onwards, we proceed by employing a "guess and verify" approach. First, we guess that both r_t and κ_t are functions of X_t and that X_t is Markov. Later we verify these conjectures.

Under the conjecture that both r_t and κ_t are functions of X_t and that X_t is Markov, the homogeneity of the recursive preferences in Equation (2) implies that there exist a pair of appropriate functions $g^i(X_t)$, $i \in \{A, B\}$, such that the time-t value function of an agent of type i born at time $s \leq t$ is given by

$$V_{t,s}^i = \frac{\left(\widetilde{W}_{t,s}^i\right)^{1-\gamma^i}}{1-\gamma^i} g^i(X_t)^{\frac{(1-\gamma^i)(\alpha^i - 1)}{\alpha^i}}. \tag{47}$$

$\widetilde{W}_{t,s}^i$ denotes the total wealth of the agent given by the sum of her financial wealth and the net present value of her earnings: $\widetilde{W}_{t,s}^i \equiv W_{t,s}^i + E_t \int_t^\infty \frac{\tilde{\xi}_u}{\tilde{\xi}_t} y_{u,s} du$. Using (47) along with the first order condition for optimal consumption $V_W = f_c$ gives $c^i_{t,s} = \widetilde{W}^i_{t,s} g^i(X_t)$. Using this last identity inside (47) and re-arranging gives

$$V^i_{t,s} = \frac{c^{1-\gamma^i}_{t,s}}{1-\gamma} g(X_t)^{-\frac{1-\gamma^i}{\alpha^i}}. \tag{48}$$

Combining Equations (48) and (46) gives

$$\left(\frac{(1-\gamma^i)V^i_{u,s}}{(1-\gamma^i)V^i_{t,s}}\right)^{\frac{1}{1-\gamma^i}} \left(\frac{g(X_u)}{g(X_t)}\right)^{\frac{1}{\alpha^i}} = e^{\frac{1}{1-\alpha^i}\int_t^u f^i_V(w)dw} \left(\frac{(1-\gamma)V^i_{u,s}}{(1-\gamma)V^i_{t,s}}\right)^{\Xi^i_4} \left(\frac{\tilde{\xi}_u}{\tilde{\xi}_t}\right)^{\frac{1}{\alpha^i-1}}. \tag{49}$$

Using the definition of f and Equation (48), we obtain

$$f^i_V(t) = \Xi^i_1 g^i(X_t) + \Xi^i_3 \tag{50}$$

Equation (50) implies that $f^i_V(t)$ is exclusively a function of X_t. Hence Equation (49) implies that $\frac{(1-\gamma^i)V^i_{u,s}}{(1-\gamma^i)V^i_{t,s}}$ is independent of s. In turn, Equation (46) implies that $\frac{c^i_{u,s}}{c^i_{t,s}}$ is independent of s. Motivated by these observations, henceforth we use the simpler notation $\frac{(1-\gamma^i)V^i_u}{(1-\gamma^i)V^i_t}$ and $\frac{c^i_u}{c^i_t}$ instead of $\frac{(1-\gamma^i)V^i_{u,s}}{(1-\gamma^i)V^i_{t,s}}$ and $\frac{c^i_{u,s}}{c^i_{t,s}}$, respectively.

Solving for $\frac{(1-\gamma^i)V^i_u}{(1-\gamma^i)V^i_t}$ from Equation (49) and applying Ito's Lemma to the resulting equation gives

$$d\left((1-\gamma^i)V^i_u\right) = \mu^i_V du + \sigma^i_V (1-\gamma^i)V^i_u dB_u, \tag{51}$$

where[42]

$$\sigma^i_V \equiv \frac{1-\gamma^i}{\gamma^i}\kappa_t - \frac{1}{\gamma^i \Xi^i_2}\frac{g^{i\prime}}{g^i}\sigma_X \tag{52}$$

$$\mu^i_V \equiv -\left(1-\gamma^i\right) f^i(c^i_u, V^i_u). \tag{53}$$

[42]Equation (53) follows from the definition of V, which implies

$$\left(1-\gamma^i\right) V^i_{t,s} + \int_s^t \left(1-\gamma^i\right) f^i\left(c_{u,s}, V_{u,s}\right) du = E_t \int_s^\infty \left(1-\gamma^i\right) f^i\left(c_{u,s}, V_{u,s}\right) du.$$

From the definition of X_t we obtain

$$X_t Y_t = \int_{-\infty}^{t} \nu^A \pi e^{-\pi(t-s)} c_{t,s}^A ds$$
$$= \int_{-\infty}^{t} \nu^A \pi e^{-\pi(t-s)} c_{s,s}^A e^{\frac{1}{1-\alpha^A}\int_s^t f_v^A(w)dw} \left(\frac{(1-\gamma^A)V_t^A}{(1-\gamma^A)V_s^A}\right)^{\Xi_4^A} \left(\frac{\tilde{\xi}_t}{\tilde{\xi}_s}\right)^{\frac{1}{\alpha^A-1}} ds \tag{54}$$

Applying Ito's Lemma to both sides of Equation (54), using (51), equating the diffusion and drift components on the left- and right-hand side, and simplifying gives

$$\frac{\sigma_X}{X_t} + \sigma_Y = \Xi_4^A \sigma_V^i - \frac{\kappa_t}{\alpha^A - 1} \tag{55}$$
$$\mu_X + X_t\mu_Y + \sigma_X\sigma_Y = \nu^A \pi \beta_t^A - \pi X_t + \frac{r_t - \rho}{1-\alpha^A} X_t + n^A X_t, \tag{56}$$

where

$$n^i \equiv \left(\frac{q^i(q^i-1)}{2}\kappa_t^2 + \frac{\Xi_4^i(\Xi_4^i - 1)}{2}\left(\sigma_V^i\right)^2 - q^i \Xi_4^A \kappa_t \sigma_V^i\right), \tag{57}$$

and $q^i \equiv \frac{1}{\alpha^i - 1}$, for $i \in \{A, B\}$. Similarly, applying Ito's Lemma to both sides of the good-market clearing equation

$$Y_t = \int_{-\infty}^{t} \nu^A \pi e^{-\pi(t-s)} c_{t,s}^A ds + \int_{-\infty}^{t} \nu^B \pi e^{-\pi(t-s)} c_{t,s}^B ds, \tag{58}$$

using (51), equating the diffusion and drift components on the left- and right-hand sides and combining with (55) and (56) leads to Equations (39) and (41). Using Equations (39) inside (55) gives (37), while (38) follows from (56). Finally, Equation (42) follows from (57) after simplifying.

The remainder of the proof shows that $\beta_{t,s}^i$, $g_{t,s}^i$ are indeed functions of X_t and shows how to obtain these functions after solving appropriate ordinary differential equations. To this end, we assume that $G(u) = B_1 e^{-\delta_1 u} + B_2 e^{-\delta_2 u}$, and define

$$\phi^j(X_t) \equiv B_j \omega \overline{h} E_t \int_t^\infty e^{-(\pi+\delta_j)(u-t)} \frac{\xi_u}{\xi_t}\frac{Y_u}{Y_{t,}} du, \tag{59}$$

so that an agent's net present value of earnings at birth is given by $Y_s\left[\sum_{j=1}^2 \phi^j(X_s)\right]$. Next notice that Equation (59) implies

$$e^{-(\pi+\delta_j)t}Y_t\xi_t\phi^j\left(X_t\right)+B_j\omega\overline{h}\int_s^t e^{-(\pi+\delta_j)u}\xi_u Y_u\,du \quad = \quad B_j\omega\overline{h}E_t\int_s^\infty e^{-(\pi+\delta_j)u}\xi_u Y_u\,du. \tag{60}$$

Applying Ito's Lemma to both sides of Equation (60), setting the drifts equal to each other, and using the fact that the right hand side of the above equation is a local martingale with respect to t (so that its drift is equal to zero) results in the differential equation

$$0 \quad = \quad \frac{\sigma_X^2}{2}\frac{d^2\phi^j}{dX^2}+\frac{d\phi^j}{dX}\left(\mu_X+\sigma_X(\sigma_Y-\kappa)\right)+\phi^j\left(\mu_Y-r-\pi-\delta_j-\sigma_Y\kappa\right)+B_j\omega\overline{h}, \tag{61}$$

where $j=1,2$. A similar reasoning allows us to obtain an expression for $g^i\left(X_t\right)$, where $i\in A,B$. Since at each point in time, an agent's present value of consumption has to equal her total wealth, we obtain

$$\frac{1}{g^i\left(X_t\right)}=\frac{\widetilde{W}^i_{t,s}}{c^i_{t,s}}=E_t\int_t^\infty\frac{\tilde{\xi}_u}{\tilde{\xi}_t}\frac{c^i_u}{c^i_t}du=E_t\int_t^\infty\frac{\tilde{\xi}_u}{\tilde{\xi}_t}\frac{c^i_u}{c^i_t}du. \tag{62}$$

Using (46) and (48) gives

$$\frac{c^i_u}{c^i_t} \quad = \quad e^{\frac{1}{\gamma^i}\int_t^u\left(\Xi^i_1 g(X_w)+\Xi^i_3\right)dw}\left(\frac{g^i(X_u)}{g^i(X_t)}\right)^{\frac{-\Xi^i_1}{\gamma^i}}\left(\frac{\tilde{\xi}_u}{\tilde{\xi}_t}\right)^{-\frac{1}{\gamma^i}}. \tag{63}$$

Using (63) inside (62) and re-arranging implies that

$$g^i\left(X_t\right)^{-1-\frac{\Xi^i_1}{\gamma^i}}\left(\tilde{\xi}_t\right)^{\frac{\gamma^i-1}{\gamma^i}}e^{\frac{1}{\gamma^i}\int_0^t\left(\Xi^i_1 g^i(X_w)+\Xi^i_3\right)dw}+\int_s^t\left(\tilde{\xi}_u\right)^{\frac{\gamma^i-1}{\gamma^i}}e^{\frac{1}{\gamma^i}\int_s^u\left(\Xi^i_1 g^i(X_w)+\Xi^i_3\right)dw}g^i(X_u)^{-\frac{\Xi^i_1}{\gamma^i}}du$$

is a local martingale. Applying Ito's Lemma and setting the drift of the resulting expression equal to zero gives

$$\begin{aligned} 0 \quad = \quad & \frac{\sigma_X^2}{2}M^i_1\left((M^i_1-1)\left(\frac{\frac{dg^i}{dX_t}}{g^i}\right)^2+\frac{\frac{d^2g^i}{dX_t}}{g^i}\right)+M^i_1\frac{\frac{dg^i}{dX_t}}{g^i}\left(\mu_X-M^i_2\sigma_X\kappa\right)+ \\ & \left(\frac{\kappa^2\left(X_t\right)}{2}M^i_2(M^i_2-1)-M^i_2\left(r\left(X_t\right)+\pi\right)-M^i_1 g^i+\frac{\Xi^i_3}{\gamma^i}\right), \end{aligned} \tag{64}$$

where $M_1^i = -1 - \frac{\Xi_1^i}{\gamma^i}$ and $M_2^i = \frac{\gamma^i - 1}{\gamma^i}$. Since both $g_{s,t}^i$ for $i \in \{A, B\}$ and ϕ^j for $j = 1, 2$ are functions of X_t, so is $\beta^i(X_t)$, which is by definition equal to $\beta^i(X_t) = g^i(X_t)\left[\sum_{j=1}^2 \phi^j(X_s)\right]$. The fact that $g_{s,t}^i$ and β_t^i are functions of X_t verifies the conjecture that X_t is Markovian and that r_t and κ_t are functions of X_t, which implies that the value functions of agents $i \in A, B$ take the form (47). ■

Proof of Propositions 1 and 2, and Corollary 1. Proposition 2 is a special case of Proposition 4 with $1 - \alpha^i = \gamma^i$ for $i \in \{A, B\}$. Similarly, Corollary 1 is a special case given by $\gamma^A = \gamma^B$.[43] Finally, when all agents are identical, then $\mu_X = \sigma_X = 0$ for all X_t, $g^i(X_t)$ and $\phi^j(X_t)$ for $i \in \{A, B\}$, $j = 1, 2$ are constants, and hence Equation (41) becomes (9), while (39) becomes $\kappa = \gamma\sigma_Y^2$. Furthermore Equation (64) becomes an algebraic equation with solution

$$g = \pi + \frac{\rho}{1-\alpha} - \frac{\alpha}{1-\alpha}\left(r + \frac{\gamma}{2}\sigma_Y^2\right), \tag{65}$$

while the present value of an agent's earnings at birth, divided by Y_s, is equal to

$$\omega\overline{h}E_s\int_s^\infty G(u-s)\frac{\xi_u}{\xi_s}\frac{Y_u}{Y_{s,}}\,du = \omega\overline{h}\left(\int_0^\infty G(u)e^{-(r+\pi+\gamma\sigma_Y^2-\mu_Y)u}du\right). \tag{66}$$

Combining (65) with (66) and the definition of β leads to (10). ■

Proof of Lemma 1. Let $\overline{r} \equiv \rho + (1-\alpha)\mu_Y - \gamma(2-\alpha)\frac{\sigma_Y^2}{2}$ denote the interest rate in the economy featuring an infinitely lived agent and also let $r^* \equiv \mu_Y - \gamma\sigma_Y^2$. Note that $\chi > \pi$ implies that $\overline{r} > r^*$. We first show that condition (13) implies the two inequalities

$$0 > \rho + (1-\alpha)\left[\mu_Y + \pi\left(1 - \beta(\overline{r})\right)\right] - \gamma(2-\alpha)\frac{\sigma_Y^2}{2} - \overline{r}, \tag{67}$$

$$0 < \rho + (1-\alpha)\left[\mu_Y + \pi\left(1 - \beta(r^*)\right)\right] - \gamma(2-\alpha)\frac{\sigma_Y^2}{2} - r^*. \tag{68}$$

When preferences are homogeneous, $\mu_X = \sigma_X = 0$, $\alpha^A = \alpha^B = \alpha$, and $\gamma^A = \gamma^B = \gamma$ so that the differential equation (64) become $g = \left(\frac{\alpha}{\alpha-1}\right)\frac{\gamma}{2}\sigma^2 + (r+\pi)\left(\frac{\alpha}{\alpha-1}\right) - \frac{\rho+\pi}{\alpha-1}$.

Using the definition of $\overline{r}$ and $\beta(\overline{r})$ on the right hand side of (67), and simplifying gives $0 > (1-\alpha)\pi\left[1 - \chi\omega\overline{h}\left(\int_0^\infty G(u)e^{-\chi u}du\right)\right]$, which is implied by (13). Similarly, using the definition of r^* and $\beta(r^*)$ inside (68) and simplifying gives $0 < \left[\rho + \pi - \alpha(\mu_Y + \pi - \gamma\frac{\sigma_Y^2}{2})\right](1-\omega)$, which is implied by the first condition in the Lemma. Given the inequalities (67) and (68), the intermediate

[43] Assuming existence of an equilibrium, the existence of a steady state follows from $\sigma_X = 0$ for all X_t, along with $\mu_X(0) = \pi\nu^A\beta^A(0) > 0$, $\mu_X(1) = -\pi\nu^B\beta^B(1) < 0$, and the intermediate value theorem.

value theorem implies that there exists a root of Equation (9) in the interval $(r^*, \overline{r})$. Accordingly, $\beta > 1$. ■

Proof of Lemma 2. When $\gamma^A = \gamma^B = \gamma$, x_2 is a function of x_1. Therefore, letting $Z \equiv \left[E(\varepsilon_2)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$, and simplifying, Equation (29) becomes

$$\left(\frac{x_2}{x_1}\right)^{\alpha^A-1} = Z^{\alpha^B-\alpha^A}\left(\frac{1-x_2}{1-x_1}\right)^{\alpha^B-1} \tag{69}$$

We show first that an implication of (69) is that when $\alpha^B > \alpha^A$, then $\frac{x_2}{x_1} < 1$. To show this, suppose (counterfactually) that $\alpha^B > \alpha^A$, but $\frac{x_2}{x_1} > 1$. Since $Z > 1$, Equation (69) implies

$$\frac{\left(\frac{x_2}{x_1}\right)^{\alpha^A-1}}{\left(\frac{1-x_2}{1-x_1}\right)^{\alpha^B-1}} = Z^{\alpha^B-\alpha^A} > 1. \tag{70}$$

The supposition $\frac{x_2}{x_1} > 1$ implies $\frac{1-x_2}{1-x_1} < 1$. Since $\alpha^A - 1 < 0$, $\alpha^B - 1 < 0$, $\left(\frac{x_2}{x_1}\right)^{\alpha^A-1} < 1$ and $\left(\frac{1-x_2}{1-x_1}\right)^{\alpha^B-1} > 1$, so that $\left(\frac{x_2}{x_1}\right)^{\alpha^A-1} < \left(\frac{1-x_2}{1-x_1}\right)^{\alpha^B-1}$, a contradiction to (70). Therefore it must be the case that $\alpha^B > \alpha^A$ implies $\frac{x_2}{x_1} < 1$. By a similar argument $\alpha^B < \alpha^A$, implies $\frac{x_2}{x_1} > 1$. This proves part *(i)*.

Taking logarithms on both sides of (69) and applying the implicit function theorem gives

$$\frac{dx_2}{dx_1} = \frac{\left(\alpha^A-1\right)\frac{1}{x_1} + \left(\alpha^B-1\right)\frac{1}{1-x_1}}{\left(\alpha^B-1\right)\frac{1}{1-x_2} + \left(\alpha^A-1\right)\frac{1}{x_2}}. \tag{71}$$

Using (71) and simplifying gives

$$\frac{d\left(\frac{x_2}{x_1}\right)}{dx_1} = \frac{\frac{dx_2}{dx_1}x_1 - x_2}{x_1^2} = \frac{\frac{x_1}{1-x_2}\left(\frac{1-x_2}{1-x_1} - \frac{x_2}{x_1}\right)}{\left(\frac{1}{1-x_2} + \frac{\alpha^A-1}{\alpha^B-1}\frac{1}{x_2}\right)x_1^2} = \frac{\frac{x_1}{1-x_2}\frac{x_1-x_2}{x_1(1-x_1)}}{\left(\frac{1}{1-x_2} + \frac{\alpha^A-1}{\alpha^B-1}\frac{1}{x_2}\right)x_1^2}. \tag{72}$$

The denominator of (72) is always positive, while the numerator is negative if and only if $x_2 > x_1$ Since $x_2 > (<)x_1$ whenever $\alpha^B < (>)\alpha^A$, it follows that $\frac{d\left(\frac{x_2}{x_1}\right)}{dx_1}$ is positive (negative) whenever $\alpha^B > (<)\alpha^A$. This proves part *(ii)*.

Finally, part *(iii)* holds by symmetry. ■

Proof of Lemma 3. When $\gamma^A = \gamma^B$, then $g^A(x_1) = \left[1 + \mathrm{E}_1\left(MRS_2^i \times \frac{c_2^i}{c_1^i}\right)\right]^{-1} = \left(1 + \beta\left(\frac{x_2}{x_1}\right)^{\alpha^A} Z^{\alpha^A}\right)^{-1}$, and hence $-\frac{1}{\alpha^A}\frac{g^{A\prime}}{g^A} = g^A\beta\left(\frac{x_2}{x_1}\right)^{\alpha^A-1} Z^{\alpha^A}\frac{d\left(\frac{x_2}{x_1}\right)}{dx_1}$. Accordingly $-\frac{1}{\alpha^A}\frac{g^{A\prime}}{g^A}$ has the same sign as $\frac{d\left(\frac{x_2}{x_1}\right)}{dx_1}$. Similarly $-\frac{1}{\alpha^B}\frac{g^{B\prime}}{g^B}$ has the same sign as $\frac{d\left(\frac{1-x_2}{1-x_1}\right)}{dx_1}$. Therefore $-\frac{1}{\alpha^A}\frac{g^{A\prime}}{g^A}$ and $-\frac{1}{\alpha^B}\frac{g^{B\prime}}{g^B}$ are positive (negative) whenever $\alpha^B > (<)\alpha^A$.

By continuity of these derivatives in γ^A and γ^B we obtain, for $\gamma^B - \gamma^A$ sufficiently close to zero,

$$\alpha^A < (>)\alpha^B \Longrightarrow -\frac{1}{\alpha^i}\frac{g^{i\prime}}{g^i} > (<)0 \text{ for both } i \in \{A, B\}. \tag{73}$$

Equating (30) and (31), solving for x_1' and using the definition of $\Gamma(x_1)$ gives

$$x_1' = \frac{x_1\left(\Gamma(x_1) - \gamma^A\right)}{\frac{\Gamma(x_1)}{\gamma^B}x_1(1-x_1)\left[\frac{1-\gamma^A-a^A}{\alpha^A}\frac{g^{A\prime}}{g^A} - \frac{1-\gamma^B-a^B}{\alpha^B}\frac{g^{B\prime}}{g^B}\right] + \gamma^A}\sigma_Y \tag{74}$$

When $\gamma^A = \gamma^B = \gamma$ and $\alpha^A = \alpha^B = \alpha$, the denominator of (74) is positive since $g^{A\prime} = g^{B\prime}$. The continuity of $g^{A\prime}$ and $g^{B\prime}$ implies the existence of $\Delta\alpha(\alpha,\gamma)$ and $\Delta\gamma(\alpha,\gamma)$ such that if $|\alpha^A - \alpha^B| < \Delta\alpha$ and $\gamma^B - \gamma^A < \Delta\gamma$, both the numerator and denominator of (74) are positive and hence $x_1' > 0$.

Both $\Delta\alpha$ and $\Delta\gamma$ can be chosen to depend continuously on their arguments. Consequently, if (α,γ) are restricted to a compact set, then there exist constants $\Delta\bar{\alpha}$ and $\Delta\bar{\gamma}$ such that, if $|\alpha^A - \alpha^B| < \Delta\bar{\alpha}$ and $|\gamma^A - \gamma^B| < \Delta\bar{\gamma}$, then (73) holds and $x_1' > 0$, implying the statements *(i)* and *(ii)* of the Lemma. ■

Proof of Proposition 4. We start by taking two arbitrary values of α^A and α^B and throughout we let $\gamma^A = \gamma^B - \varepsilon$. For $\kappa(X_t)$ given by (39), we let $Z(X_t;\varepsilon) \equiv \kappa(X_t) - \Gamma(X_t)\sigma_Y$. Clearly $Z(\bar{X};0) = 0$. To prove the proposition, it suffices to show that $\left.\frac{dZ(\bar{X};\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} > 0$. Direct differentiation, along with the fact that $\sigma_X = 0$ when $\varepsilon = 0$, gives

$$\left.\frac{dZ(\bar{X};\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \sum_{i\in\{A,B\}}\omega^i(\bar{X})\left(\frac{1-\gamma^i-a^i}{\alpha^i}\right)\frac{g^{i\prime}(\bar{X},0)}{g^i(\bar{X},0)} \times \left(\left.\frac{d\sigma_X(\bar{X};\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}\right).$$

To determine the sign of $\frac{g^{i\prime}(\bar{X},0)}{g^i(\bar{X},0)}$, we set $\gamma^A = \gamma^B = \gamma$, so that both $\sigma_X = 0$ and $\kappa = \gamma\sigma$.

Equation (64) can now be written as

$$0 = M_1^i \frac{g^{i\prime}}{g^i}\mu_X + \left[\frac{(\gamma\sigma)^2}{2}M_2^i\left(M_2^i-1\right) - M_2^i\left(r\left(X_t\right)+\pi\right) + \frac{\Xi_3^i}{\gamma} - M_1^i g^i\right]. \tag{75}$$

Differentiating (75) with respect to X and evaluating the result at $X=\bar{X}$ using $\mu_X=0$ yields

$$\frac{g^{i\prime}}{g^i}M_1^i\left(\mu_X^\prime - g^i\right) = M_2^i r^\prime, \tag{76}$$

where primes denote derivatives with respect to X. Re-arranging (76) and using the definitions of M_1^i and M_2^i gives

$$\frac{g^{i\prime}}{g^i} = \frac{M_2^i}{M_1^i}\left(\frac{r^\prime}{\mu_X^\prime - g^i}\right) = \frac{\alpha^i}{1-\alpha^i}\frac{r^\prime}{\mu_X^\prime - g^i}. \tag{77}$$

A similar computation starting from Equation (61) and using $\phi^j\left(\bar{X}\right) = \frac{B_j\omega\bar{h}}{r+\pi+\delta_j+\gamma\sigma^2-\mu_Y}$ yields

$$\frac{\phi^{j\prime}}{\phi^j} = -\frac{1}{\frac{B_j\omega\bar{h}}{\phi^j}-\mu_X^\prime}r^\prime. \tag{78}$$

The fact that $\overline{X}$ is a stable steady state implies that $\mu_X^\prime \leq 0$, and accordingly Equation (77) implies that $\frac{1}{\alpha}\frac{g^{i\prime}}{g^i}$ has the opposite sign from $r^\prime$. Similarly, Equation (78) implies that $\frac{\phi_j^\prime}{\phi_j}$ has the opposite sign[44] from $r^\prime$.

We next show that if $\gamma^A=\gamma^B=\gamma$ and $\alpha^B>\alpha^A$, then $r^\prime\left(\bar{X}\right)>0$. We proceed by supposing the contrary: $\gamma^A=\gamma^B=\gamma$ and $\alpha^B>\alpha^A$, but $r^\prime\leq 0$. Differentiating equation (41) with respect to X_t and evaluating the resulting expression around $X_t=\overline{X}$, we obtain

$$r^\prime = \frac{1}{\Theta\left(\bar{X}\right)}\left\{\left(\frac{1}{1-\alpha^B}-\frac{1}{1-\alpha^A}\right)[r-\rho]-n^A+n^B\right\} - \pi\frac{1}{\Theta\left(\bar{X}\right)}\sum_i v^i\left(\beta^i\right)^\prime. \tag{79}$$

The definition of $n^i\left(X_t\right)$ in Equation (42) along with $\gamma^A=\gamma^B=\gamma$ and $\sigma_X=0$ gives

$$n^A-n^B = \left[\frac{2-\alpha^A}{\left(1-\alpha^A\right)}-\frac{2-\alpha^B}{\left(1-\alpha^B\right)}\right]\frac{\kappa^2\left(\bar{X}\right)}{2\gamma} = \gamma\left(\frac{1}{1-\alpha^A}-\frac{1}{1-\alpha^B}\right)\frac{\sigma_Y^2}{2}. \tag{80}$$

[44] Note that B_j and ϕ^j have the same sign.

Furthermore, noting that $\beta^i = g^i \sum_{j=1,2} \phi^j$ and using (77) and (78) leads to

$$\pi \sum_{i\in\{A,B\}} v^i \left(\beta^i\right)' = -r' \left[\pi \sum_{i\in\{A,B\}} v^i \left(\frac{\alpha^i}{1-\alpha^i} \frac{1}{g^i - \mu_X'} g^i \sum_{j=1,2} \phi_j + g^i \sum_{j=1,2} \phi_j \frac{1}{\frac{B_j \omega \overline{h}}{\phi^j} - \mu_X'}\right)\right]$$
$$\leq -r' \left[\pi \sum_{i\in\{A,B\}} v^i \left(\frac{(\alpha^i)^+}{1-\alpha^i} \sum_{j=1,2} \phi_j + g^i \sum_{j=1,2} \frac{\phi_j^2}{B_j \omega \overline{h}}\right)\right] \quad (81)$$

The inequality (81) follows from the facts that (a) r' has been assumed to be non-positive, (b) $\mu_X' \leq 0$ and (c) $\sum_{j=1,2} \phi_j \frac{1}{\frac{B_j \omega \overline{h}}{\phi^j} - \mu_X'}$ is positive and increasing in μ_X'.[45]

Let

$$\eta \equiv \pi \sum_{i\in\{A,B\}} v^i \left(\frac{(\alpha^i)^+}{1-\alpha^i} \sum_{j=1,2} \phi_j + g^i \sum_{j=1,2} \frac{\phi_j^2}{B_j \omega \overline{h}}\right). \quad (82)$$

Using $\phi^j\left(\overline{X}\right) = \frac{B_j \omega \overline{h}}{r+\pi+\delta_j+\gamma\sigma^2-\mu_Y}$ and Equation (65), we obtain that, when $\alpha^A = \alpha^B = \alpha$,

$$\eta = \pi \left(\frac{\alpha^+}{1-\alpha} \sum_{j=1,2} \frac{\omega B_j \overline{h}}{r+\pi+\delta_j+\gamma\sigma_Y^2-\mu_Y} + \sum_{j=1,2} \frac{\omega B_j \overline{h}\left(\pi + \frac{\rho}{1-\alpha} - \frac{\alpha}{1-\alpha}\left(r+\frac{\gamma}{2}\sigma_Y^2\right)\right)}{\left(r+\pi+\delta_j+\gamma\sigma_Y^2-\mu_Y\right)^2}\right),$$

where r is given in Equation (9). We shall assume that when $\alpha^A = \alpha^B = \alpha$,

$$\eta < 1-\alpha, \quad (83)$$

which is the case, for instance, when ω is sufficiently small.[46] Using (81) and (80) inside (79) gives

$$r' \geq \frac{1}{\Theta\left(\overline{X}\right)} \left(\frac{1}{1-\alpha^B} - \frac{1}{1-\alpha^A}\right)\left[r - \rho + \gamma\frac{\sigma_Y^2}{2}\right] - \frac{\eta}{\Theta\left(\overline{X}\right)} r', \quad (84)$$

[45] To see that $\sum_{j=1,2} \phi_j \frac{1}{(\phi^j)^{-1} B_j \omega \overline{h} - \mu_X'}$ is positive, note that $\frac{1}{(\phi^1)^{-1} B_1 \omega \overline{h} - \mu_X'} = \frac{1}{r+\pi+\delta_1+\sigma_Y\kappa-\mu_Y-\mu_X'} > \frac{1}{r+\pi+\delta_2+\sigma_Y\kappa-\mu_Y-\mu_X'}$ since $\delta_1 < \delta_2$. Furthermore, since $B_1 > -B_2$, it follows that $\phi^1 > -\phi^2$. Treating μ_X' as a variable and differentiating $\sum_{j=1,2} \phi_j \frac{1}{(\phi^j)^{-1} B_j \omega \overline{h} - \mu_X'}$ with respect to μ_X' shows that $\sum_{j=1,2} \phi_j \frac{1}{(\phi^j)^{-1} B_j \omega \overline{h} - \mu_X'}$ is increasing in μ_X'.

[46] Note that an implication of Lemma 1 is that the interest rate satisfies $r > \mu_Y - \gamma\sigma_Y^2$ for any value of ω, and hence the expressions $\frac{1}{r+\pi+\delta_j+\gamma\sigma_Y^2-\mu_Y}$ and $\frac{\pi + \frac{\rho}{1-\alpha^i} - \frac{\alpha^i}{1-\alpha^i}\left(r+\frac{\gamma}{2}\sigma_Y^2\right)}{r+\pi+\delta_j+\gamma\sigma_Y^2-\mu_Y}$ must approach finite limits as ω goes to zero.

or

$$\left[\Theta\left(\overline{X}\right)-\eta\right]r' \quad \geq \quad \left(\frac{1}{1-\alpha^B}-\frac{1}{1-\alpha^A}\right)\left[r-\rho+\gamma\frac{\sigma_Y^2}{2}\right]. \tag{85}$$

When $\alpha^A=\alpha^B=\alpha$, the term $r-\rho+\gamma\frac{\sigma_Y^2}{2}$ is positive if[47]

$$\mu_Y-\frac{\gamma\sigma_Y^2}{2}+\pi \quad > \quad \omega\left(\rho+\pi\right)\frac{\sum_{j=1,2}\frac{B_j}{\delta_j+\rho-\mu+\frac{\gamma}{2}\sigma_Y^2+\pi}}{\sum_{j=1,2}\frac{B_j}{\delta_j+\pi}}, \tag{86a}$$

$$\rho+\pi\left(1-\alpha\right) \quad > \quad \alpha\left(\mu_Y-\gamma\frac{\sigma_Y^2}{2}\right) \tag{86b}$$

We note that condition (86b) is automatically satisfied when $\alpha<0$ and $\mu_Y\geq\gamma\frac{\sigma_Y^2}{2}$, while condition (86a) holds when ω is small.

When $\left|\alpha^A-\alpha^B\right|$ is small (but not zero), continuity implies that the right hand side of (85) has the same sign as $\frac{1}{1-\alpha^B}-\frac{1}{1-\alpha^A}$, which is positive. However, given the supposition that r' is non-positive and assumption (83), the left-hand side of (85) is non-positive, which is a contradiction. We therefore conclude that, when $\alpha^B>\alpha^A$, $r'>0$. A symmetric argument implies that, when $\alpha^B<\alpha^A$, $r'<0$. Recalling that r' and $\frac{1}{\alpha^i}\frac{g^{i\prime}}{g^i}$ have the same sign, it follows that the term $\sum_{i\in\{A,B\}}\omega^i\left(\bar{X}\right)\left(\frac{1-\gamma^i-a^i}{\alpha^i}\right)\frac{g^{i\prime}\left(\bar{X},0\right)}{g^i\left(\bar{X},0\right)}$ is positive when either condition 1 or condition 2 of the Lemma holds.

Consider now the term $\left.\frac{d\sigma_X\left(\bar{X};\varepsilon\right)}{d\varepsilon}\right|_{\varepsilon=0}$, which we want positive. Direct differentiation of (37) gives

$$\left.\frac{d\sigma_X\left(\bar{X};\varepsilon\right)}{d\varepsilon}\right|_{\varepsilon=0}=\frac{\bar{X}\left(\frac{d\Gamma\left(\bar{X};\varepsilon\right)}{d\varepsilon}+1\right)}{\bar{X}\left(1-\bar{X}\right)\left[\frac{1-\gamma-\alpha^A}{\alpha^A}\frac{g^{A\prime}}{g^A}-\frac{1-\gamma-\alpha^B}{\alpha^B}\frac{g^{B\prime}}{g^B}\right]+\gamma^B}. \tag{87}$$

[47] When the economy is populated by a single agent, we can replicate the steps of the proof of Lemma 1 to prove this fact. Specifically, letting $r^{**}=\rho-\frac{\gamma}{2}\sigma_Y^2$ we arrive at the conclusion that

$$0<\rho+\left(1-\alpha\right)\left[\mu_Y+\pi\left(1-\beta\left(r^{**}\right)\right)\right]-\gamma\left(2-\alpha\right)\frac{\sigma_Y^2}{2}-r^{**},$$

as long as condition (86a) holds. Similarly, setting $\overline{\overline{r}}=\rho+\left(1-\alpha\right)\left(\mu_Y+\pi\right)-\gamma\left(2-\alpha\right)\frac{\sigma_Y^2}{2}$ gives

$$0>\rho+\left(1-\alpha\right)\left[\mu_Y+\pi\left(1-\beta\left(\overline{\overline{r}}\right)\right)\right]-\gamma\left(2-\alpha\right)\frac{\sigma_Y^2}{2}-\overline{\overline{r}}.$$

as long as Equation (86b) holds. Hence there must exist a root in the interval $\left(r^{**},\overline{\overline{r}}\right)$.

The numerator on the right hand side of Equation (87) is positive, and as long as α^A and α^B are close enough, $\frac{1-\gamma-\alpha^A}{\alpha^A}\frac{g^{A\prime}}{g^A} - \frac{1-\gamma-\alpha^B}{\alpha^B}\frac{g^{B\prime}}{g^B}$ is arbitrarily close to zero, so that the denominator is also positive.

The proof can now be concluded by repeating the same arguments as in the last two paragraphs of the proof of Lemma 3. ■

Proof of Proposition 5. Using (8), applying Ito's lemma to compute $d\left(e^{-\pi s}\xi_s W_s^i\right)$, integrating, and using a transversality condition we obtain

$$W_{t,s}^i = E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t}\left(c_{u,s}^i - y_{u,s}\right) du. \tag{88}$$

The market-clearing equations for stocks and bonds imply

$$S_t = \sum_{i\in\{A,B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} \upsilon^i W_{t,s}^i ds. \tag{89}$$

Substitution of (88) into (89) gives

$$\begin{aligned} S_t = & \sum_{i\in\{A,B\}} \int_{-\infty}^t \pi e^{-\pi(t-s)} \upsilon^i \left[E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du\right] ds \\ & - \int_{-\infty}^t \pi e^{-\pi(t-s)} \left[E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} y_{u,s} du\right] ds. \end{aligned} \tag{90}$$

We can compute the first term in (90) as

$$\begin{aligned} & \sum_{i\in\{A,B\}} \upsilon^i \int_{-\infty}^t \pi e^{-\pi(t-s)} \left[E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} c_{u,s}^i du\right] ds \\ = & \sum_{i\in\{A,B\}} \upsilon^i \int_{-\infty}^t \pi e^{-\pi(t-s)} c_{t,s}^i \left[E_t \int_t^\infty e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} \frac{c_{u,s}^i}{c_{t,s}^i} du\right] ds \\ = & \sum_{i\in\{A,B\}} \upsilon^i \int_{-\infty}^t \pi e^{-\pi(t-s)} \frac{c_{t,s}^i}{g\left(X_t\right)} ds = Y_t \left[\frac{X_t}{g^A\left(X_t\right)} + \frac{1-X_t}{g^B\left(X_t\right)}\right] \end{aligned} \tag{91}$$

Similarly, using (4) and (14) we can compute the second term in (90) as

$$\begin{aligned}
&\int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left[E_t \int_t^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} y_{u,s} du \right] ds \\
&= \omega \overline{h} \int_{-\infty}^{t} \pi e^{-\pi(t-s)} \left[E_t \int_t^{\infty} e^{-\pi(u-t)} \frac{\xi_u}{\xi_t} Y_u \left(\sum_{j=1}^{2} B_j e^{-\delta_j (u-s)} \right) du \right] ds \\
&= Y_t \times \sum_{j=1}^{2} \int_{-\infty}^{t} \pi e^{-(\pi+\delta_j)(t-s)} \left[B_j \omega \overline{h} E_t \int_t^{\infty} e^{-(\pi+\delta_j)(u-t)} \frac{\xi_u}{\xi_t} \frac{Y_u}{Y_t} du \right] ds \\
&= Y_t \times \sum_{j=1}^{2} \int_{-\infty}^{t} \pi e^{-(\pi+\delta_j)(t-s)} \phi^j (X_t) \, ds \\
&= Y_t \times \sum_{j=1}^{2} \frac{\pi}{\pi+\delta_j} \phi^j (X_t) . \qquad (92)
\end{aligned}$$

Combing (91) and (92) inside (90) implies that $\frac{S_t}{Y_t}$ is a function of X_t. Since $\sigma_X(X_t) = 0$ when $\gamma^A = \gamma^B$ (Corollary 1) it follows that $\sigma_t = \sigma_Y$ when $\gamma^A = \gamma^B$. ■